Exploration 1

Triangle Sums

INTRODUCTION

Prior Knowledge

» Add, subtract, multiply and divide one- and two-digit numbers.

Learning Goals

» Experience a mathematical investigation process.
» Develop and flexibly apply problem-solving strategies.
» Analyze and extend patterns.
» Make and test mathematical conjectures.
» Justify conclusions using deductive reasoning.
» Formulate questions and generalize solutions.
» Communicate complex mathematical ideas clearly.
» Persist in solving challenging problems.

Launching the Exploration

Motivation and purpose. To students: This exploration is an extended version of an entertaining puzzle that has been around for years. Its main purpose is to introduce you to the process that mathematicians use to discover and create new mathematical knowledge. You will gather information, organize it, analyze it, make predictions, test them, try to prove (or disprove!) your conclusions, and then create new questions to explore. In the process, you will strengthen your computational, problem-solving, and mathematical reasoning skills.

Understanding the problem. Read through the first question to ensure that students understand it. Emphasize the fact that each number must be used exactly once—none of them will be left out or repeated.

Encourage students to begin the exploration using "thinking paper." This is a place where they record their ideas, calculations, conjectures, and observations—essentially everything they do—as they work. They should save this paper and use it to help them write their final copy.

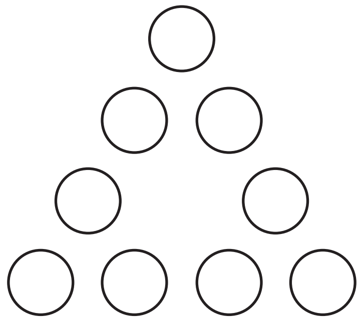
Some students may enjoy cutting out nine circles, numbering them, and moving them around as they try to solve the puzzle. However, remind them that they should still keep track of their work on the thinking paper.

NAME: ______________________________ DATE: ______________

STUDENT HANDOUT

Stage 1

1. Use each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 exactly once to fill in the circles. Make the sum of each side equal to 17. Describe your thinking strategies.



2. Find a different solution to the problem.

3. What important feature do your solutions have in common?

4. How can you see that these are the only two solutions?

Stage 2

5. Add the counting numbers 1 through 9. Then calculate $17 \cdot 3$. How does this relate to the original Triangle Sums problem? Why aren't the two answers the same?

6. Why must your two solutions have the common feature you described in Problem #3? How does this show you another way to see that there are only two solutions? Explain.

7. Fill in the circles with the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 as before. Try to make the three sides have the same sum, but this time larger than 17!

8. Why can't "Triangle Sums" be solved for a sum smaller than 17? Explain.

Stage 3

9. Suppose you have a triangle with five circles on each side instead of four. Using the numbers 1–12, what is the smallest sum that will allow a solution to the problem? Explain your thinking.

10. Solve the new problem using this sum.

11. Imagine triangles with more and more circles on each side. Describe a method for finding the smallest sum that will allow a solution, no matter how many circles are on each side of the triangle.

12. Complete a table showing the smallest sum for triangles with 3, 4, 5, 6, and 7 circles per side.

13. Discover and describe a pattern in the smallest sums. Use your pattern to predict the next three sums.

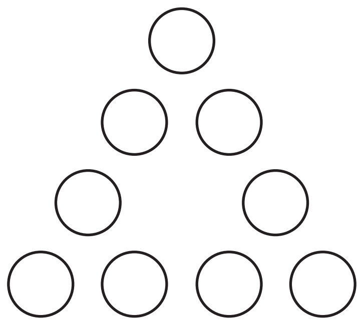
14. Think of at least three more questions to extend the "Triangle Sums" exploration.

TEACHER'S GUIDE

STAGE 1

Problem #1

1. Use each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 exactly once to fill in the circles. Make the sum of each side equal to 17. Describe your thinking strategies.



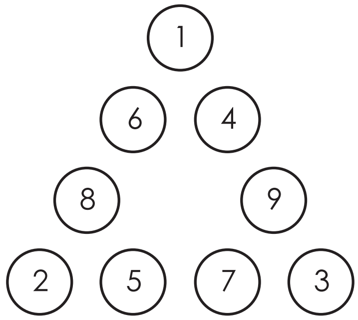
Questions and Conversations for #1

This section contains ideas for conversations, mainly in the form of questions that students may ask or that you may pose to them. Be sure to allow students to do most of the thinking and talking!

- » *What might cause the sums to be too large?* Putting large numbers on the same side of the triangle or in the corners causes a problem. Can you see why?
- » *How can you use unsuccessful attempts to move you toward a correct solution?* Rather than starting over when something doesn't work, change your earlier attempts one small step at a time, paying attention to how each change affects the sums.
- » *How many ways can you make a sum of 17 with four numbers from 1 through 9?* This might be worth investigating. It helps organize your work and limits the number of things you have to try. It might also help you to find more solutions later, and to know when you've found all of them.
- » *How can you explain your thinking process if it feels like you just tried numbers randomly until you found something that worked?* You may feel that you chose numbers randomly, but you almost certainly made choices along the way. Did you notice that your sums were too large? Did you make a plan to fix this problem? Did you keep track of things you had already tried? Did you list different ways to make sums of 17? Did you make small adjustments when you were close to a solution?

Solution for #1

A sample solution is:



Sample student responses:

- » Avoid putting large numbers together on the same side of the triangle.
- » Notice that placing large numbers in the corners makes the sums too large (because these numbers are used more than once in the sums). Try smaller numbers in the corners.
- » Use a guess, check, and revise strategy. Put numbers in randomly, add the sides, and then make adjustments by switching pairs of numbers in order to raise some sums and lower others.
- » Find sets of four numbers that add to 17 (like 1, 2, 5, 9 and 1, 2, 6, 8 for example). Then find ways to fit them together into the triangle.
- » Notice that when you find sets of four numbers that add to 17, the numbers 1, 2, and 3 appear more often than others. This means that 1, 2, and 3 should be in the corners.

Problem #2

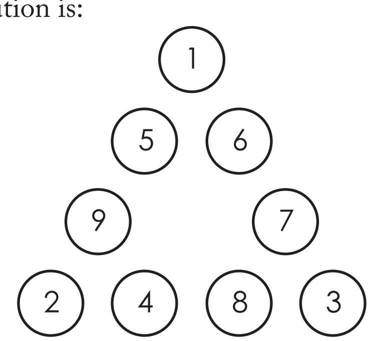
2. Find a different solution to the problem.

Questions and Conversations for #2

- » *What counts as a different solution?* You can decide, but most people agree that rotating or reflecting a solution, or interchanging two numbers in the middle of a side does not count as a new solution. This is what we mean when we say that there are only two solutions.

Solution for #2

Another sample solution is:



Problem #3

3. What important feature do your solutions have in common?

Questions and Conversations for #3

» *Where are the most important circles in the triangle? Why?* The most important circles are the ones in the corners, because each of them contributes to two of the sums.

» *Can you use your original solution to help you find a new one?* Possibly—if you pay attention to what should stay the same and what can change.

Solution for #3

In both solutions, the numbers 1, 2, and 3 are in the corners of the triangle.

Problem #4

4. How can you see that these are the only two solutions?

Questions and Conversations for #4

» *What happens to a solution if you exchange a corner number with one of the two middle numbers on any side?* It will always change one of the sums, making it greater than 17.

Solution for #4

Once you have a solution with 1, 2, and 3 in the corners, if you trade a corner number with a noncorner number (which will always be greater than 3), the sum of one or both sides touching that corner becomes greater than 17, ruining the solution! (What happens if you trade two corner numbers?)

STAGE 2

Problem #5

5. Add the counting numbers 1 through 9. Then calculate $17 \cdot 3$. How does this relate to the original Triangle Sums problem? Why aren't the two answers the same?

Questions and Conversations for #5

» *How do the answers to the two calculations compare? What causes this?* The two answers differ by 6. Why? Think about how this relates to the most important circles in the triangle.

Solution for #5

$$1+2+3+4+5+6+7+8+9=45 \qquad 17\cdot 3=51$$

The equation on the left shows the sum of all of the numbers you place in the circles. The product on the right comes from adding the sums of the three sides of the triangle. The answers aren't the same because when you add these sums, you count each corner twice.

Problem #6

6. Why must your two solutions have the common feature you described in Problem #3? How does this show you another way to see that there are only two solutions? Explain.

Questions and Conversations for #6

See Questions and Conversations for #5.

Solution for #6

The corners must account for the difference of 6 between 45 and 51. The only way you can make the corners add to 6 is to place the numbers 1, 2, and 3 in them.

Because 1, 2, and 3 are in the corners, your choices for placing the remaining numbers are limited:

» The side containing 1 and 2 needs 14 more.
» The side containing 1 and 3 needs 13 more.
» The side containing 2 and 3 needs 12 more.

There are only two ways to accomplish this using the remaining six numbers:

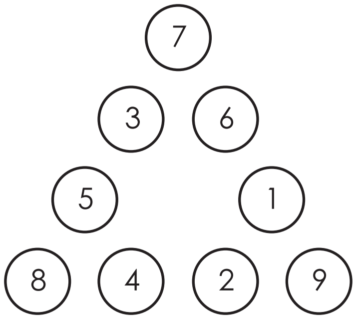
$14=5+9$	$13=6+7$	$12=4+8$
$14=6+8$	$13=4+9$	$12=5+7$

Problem #7

7. Fill in the circles with the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 as before. Try to make the three sides have the same sum, but this time larger than 17!

Solution for #7

A sample solution (this one happens to have the largest possible sum, 23):



Problem #8

8. Why can't "Triangle Sums" be solved for a sum smaller than 17? Explain.

Questions and Conversations for #8

» *What would have to happen with the corners?* They would have to have smaller numbers in them.

Solution for #8

You obtain the smallest sum by using the smallest possible numbers in the corners. Because 1, 2, and 3 are the smallest numbers available, you can't form a sum smaller than 17.

STAGE 3

Problem #9

9. Suppose you have a triangle with five circles on each side instead of four. Using the numbers 1–12, what is the smallest sum that will allow a solution to the problem? Explain your thinking.

Questions and Conversations for #9

» *What numbers must be in the corners in order to achieve the lowest sum?* As before, the corners must hold the numbers 1, 2, and 3.

» *How should three times the sum of each side relate to the sum of the counting numbers in the circles?* The values in the corners can help you answer this question.

Solution for #9

The smallest sum is 28.

$$1+2+3+4+5+6+7+8+9+10+11+12=78$$

When you make the smallest sum, you must place 1, 2, and 3 in the corners. Because the corners must still have a sum of 6, you add $78+6=84$. (This counts each corner twice.) The three sums must add to 84, so the sum for each side is $84 \div 3 = 28$.

Problem #10

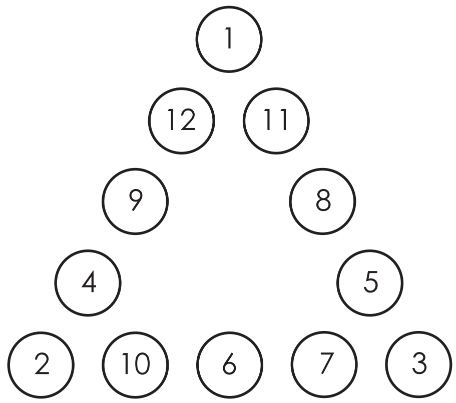
10. Solve the new problem using this sum.

Questions and Conversations for #10

See Questions and Conversations for #9.

Solution for #10

A sample solution:



Problem #11

11. Imagine triangles with more and more circles on each side. Describe a method for finding the smallest sum that will allow a solution, no matter how many circles are on each side of the triangle.

Questions and Conversations for #11

See Questions and Conversations for #9.

Solution for #11

Follow the same process we used above. Count the number of circles in the triangle. Add all of the counting numbers from 1 through this number. Add 6 to this total and then divide the result by 3.

Problem #12

12. Complete a table showing the smallest sum for triangles with 3, 4, 5, 6, and 7 circles per side.

Solution for #12

Number of Circles Per Side	Smallest Sum for a Solution
3	9
4	17
5	28
6	42
7	59

Problem #13

13. Discover and describe a pattern in the smallest sums. Use your pattern to predict the next three sums.

Questions and Conversations for #13

» *How do you get from one number to the next for the list of smallest sums in your table?* Focus on the differences between the numbers.

Solution for #13

The "smallest sum" seems to be increasing by 8, 11, 14 and then 17. The difference is increasing by 3 each time. If this pattern continues, the smallest sums will be

$59 + 20 = 79$	for 8 circles
$79 + 23 = 102$	for 9 circles
$102 + 26 = 128$	for 10 circles

Problem #14

14. Think of at least three more questions to extend the "Triangle Sums" exploration.

Questions and Conversations for #14

» *Were there any questions that arose in your mind as you were working? Is there anything you were curious about?* Answers will vary.

» *We adjusted the number of circles per side. What other things might you change?* You could change the basic shape from a triangle to something else. You could try a different operation than addition. You might try placing different kinds of numbers in the circles.

Solution for #14

See the "Further Exploration" feature at the end of this activity for some ideas.

WRAP UP

Share Strategies

Have students share their strategies and compare results.

Summarize

Answer any remaining questions that students have. Summarize a typical mathematical exploration process:

» ask a question;
» explore—gather information and search for strategies;
» organize and analyze the information, look for patterns;
» make conjectures or predictions;
» test the conjectures;
» prove the conjectures or try to understand why they are (or are not) true; and
» reflect on what you've learned and think of new questions to ask.

Help students see how "Triangle Sums" fits this pattern of mathematical exploration. Encourage them to remain aware of this process in future explorations. At the same time, make sure they understand that no single list can capture the true nature of what it means to "do mathematics." You will sometimes leave out some of these steps, incorporate different ones, or do them in a different order.

Further Exploration

Ask students to think of ways to continue or extend this exploration. Here are some possibilities:

» Are there patterns in the largest possible sums for solutions? How do they relate to the smallest sums?
» Is it possible to solve the problem for each whole number between the smallest and largest possible sums?
» What happens if you use squares instead of triangles?
» What happens if the sides must have the same product?
» How many solutions will the original problem have if it does count as a new solution when you rotate or reflect the original triangle or exchange the numbers in the middle of a side? (Answer: 96)
» Can you find a formula for the relationship in the table from Problem #12? (Sample answer: $y = \frac{1}{2}\left(3x^2 - 5x + 6\right)$ where x represents the number of circles per side, and y stands for the smallest sum.) This will be a big challenge because most students won't know formal algebraic procedures, but never underestimate what a determined student can accomplish! Those who do have more background in algebra might try to derive the formula from the problem situation.

Exploration 2

Torran Math

INTRODUCTION

Prior Knowledge

- » Understand how place value is used to represent numbers.
- » Understand the role of place value in procedures for whole number addition, subtraction, multiplication, and division.

Learning Goals

- » Deepen understanding of place value by exploring a system that groups by a number other than 10.
- » Develop, describe, and justify procedures for translating between two place value systems.
- » Analyze and extend counting patterns in a new place value system.
- » Understand the difference between numbers (ideas) and numerals (symbols).
- » Communicate complex mathematical ideas clearly.
- » Persist in solving challenging problems.

Launching the Exploration

Motivation and purpose. To students: One of the best ways to better understand your own language is to learn someone else's. In the same way, if you want to gain a deeper understanding of your own numeration system, it helps to study a different system! To do this, you will visit the imaginary planet, Torr, where they group everything by fours instead of tens. As you investigate the Torran way of writing numerals, pay close attention to the similarities and differences between our systems.

Understanding the problem. Ask students to read the first page of the activity very carefully. Have them discuss what they've learned and what they are still trying to understand. (They should not feel that they have to understand everything right now.) Encourage students to refer to the first page whenever they feel confused or get stuck.

Discuss the distinction between *numbers* and *numerals*. Tell students to stay focused on this distinction throughout the exploration.

Don't explain anything beyond the content of the introductory page. Especially do not teach explicit procedures for translating between the Earth and Torran numeration systems. Students will develop their own strategies based upon their own understandings.

Teacher's Note. Even though this exploration is about different number *base* systems, notice that we never use the word *base*. This is to encourage students to think as independently as possible and avoid the temptation to "look it up." You might like to introduce the term at the end of the exploration.