Intrepid Math

Challenging Common Core Problems

Grades 4 – 6
Set 2

from 5280 Math
by Jerry Burkhart

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Intrepid Math
Challenging Common Core Problems
Quick Start Guide

1. **Prepare**
   Choose one problem.
   Make copies.
   Gather materials.
   Familiarize yourself with the problem.

2. **Introduce**
   Distribute the problem, materials, and *thinking paper**.
   Give directions.
   Establish expectations.
   Set an estimated time frame.

3. **Support**
   Facilitate discussion or check in with individuals and groups.
   Help students clarify their thoughts.
   Ask guiding questions.
   Encourage and motivate students through the tough spots.
   Determine when students have reached a stopping point.

4. **Wrap up**
   Gather completed work with explanations.
   Acknowledge students' effort and progress.
   Summarize what was learned.
   Assess student work.

Advice for students
Plan on spending a few days or more on many of the problems.
The problems are challenging. Expect to get stuck and to make some mistakes.
Take your time. It is more important to learn new things than to finish the entire problem.

*See the FAQs to learn about thinking paper and strategies for teaching with the problems.

Name ________________________________

Problem 1

You have a 10-centimeter piece of string.

Part 1: Can you bend it into a shape that has an area of 6 square cm?

Part 2: Can you bend it into a shape that has an area of 2.25 square cm?

Part 3: Can you bend it into a shape that has an area of 6.25 square cm?

Name ________________________________

Problem 1: *Testing the Waters*

You have a 20-centimeter piece of string. Show how to bend it into at least six shapes, each having a different area. Look for patterns and describe them.

Name ________________________________

Problem 1: *Diving Deeper*

Choose one or more questions to answer, or think of your own new questions to explore.

Continue to explore new ways to bend the 10-centimeter string.

1. Make the area equal or nearly equal to 1 square cm.
2. Make the area greater than 7 square cm.
3. Make a *non-rectangular* figure with an area of 6 square cm.

Problem 1
Teacher's Guide

Topics

Area, perimeter, and the relationship between them; decimals; distributive property

Materials

Graph paper

The grade level notes below are a rough guide for working with different age groups. Read through all three grades to get a feel for the likely range of students' thinking. Their approaches may vary a lot, so never feel limited to a particular grade level. Observe your students' ideas and strategies to assess their thinking, and build from there.

Grade 4 Common Core Standards: 4.MD.A

Students should use graph paper to explore and test their ideas. (They may think of square *units* instead of square centimeters if they prefer.) Once they find solutions, they can record them on the handout by sketching them and labeling the side lengths or by cutting the shapes out from graph paper and attaching them to the handout.

Before beginning Part 2, make sure that students know that 0.25 represents ¼. *Do not press fourth graders to use the area formula for rectangles*. The important thing form them is to experiment and to *see* and count the square units inside the shapes, especially the half- and quarter-squares.

Students learn from this problem that shapes with the same perimeter do not always have the same area. If they are ready to continue exploring area-perimeter connections, try the Testing the Waters questions or look at the grade 5 notes.

Grade 5 Common Core Standards: 5.NBT.A 5.NBT.B 5.NF.B.4.b

Older students should still make drawings on graph paper. If they know how to multiply decimals, they may multiply length and width, but they should connect their procedures to their drawings. Some students may use their pictures to *discover* strategies for multiplying decimals. See the "Summary" sections of the Solutions.

Students may further explore connections between area and perimeter by using the Testing the Waters problem or by trying to make shapes with smaller and smaller areas (keeping the perimeter at 10). Guide them to notice and explain that the small-area rectangles tend to be "long and skinny," while the larger area shapes tend to be more compact.

Grade 6 Common Core Standards: 6.NS.B.3 6.G.A.1

Students in grade 6 may finish sooner and spend more of their time on the Diving Deeper questions. These questions involve estimation or guess/test/revise strategies (unless students know about the Pythagorean theorem or quadratic equations). Students may use calculators on these problems.

Solutions begin on page 37.

Name ______________________________

Problem 2

When Janine eats 1/3 of her chocolate bar and Marco eats 1/5 of his, each person has the same amount left. Janine's bar originally had 2.5 ounces more chocolate than Marco's. How many ounces did each person start with? How many ounces do they have left?

Name ________________________________

Problem 2: *Testing the Waters*

Janine's chocolate bar has 4 ounces more chocolate than Marco's. When Janine eats 2/3 of her bar and Marco eats 1/2 of his, each person has the same amount. How many ounces of chocolate does each person have now? How much did they start with?

Name ______________________________

Problem 2: *Diving Deeper*

Choose one or more questions to answer, or think of your own new questions to explore.

Janine’s intial amount: J   Marco’s intial amount: M

1. Graph the relationship between J and M in the first sentence.
2. Graph the relationship between J and M in the second sentence.
3. Show the solution to the original problem on your graphs.
4. Write an equation for each relationship.
5. Combine your equations into a single equation containing only one of M or J.
6. Solve your equation.

Problem 2
Teacher's Guide

Topics

Fraction of a number; patterns; proportional reasoning

Materials

Graph paper (recommended)

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Grade 4 Common Core Standards: 4.NF.A

Students of all ages should be encouraged to draw pictures and diagrams of the fractions on graph paper.

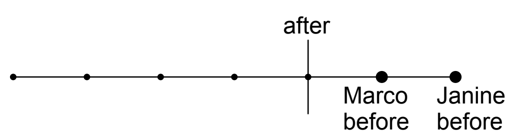
Expect fourth graders to rely pretty heavily on a guess/test/revise approach. Since the problem is complex, some of them may need help getting started. Suggest that they pick a simple amount like 3 ounces for Janine. It may be easier to focus on the difference of 2.5 (the second sentence) before they start dealing the fractions. (See Strategy #1 in the Solutions.) If they are having trouble keeping track of their work, suggest that they make tables. Some students may see patterns in their tables.

Grade 5 Common Core Standards: 5.OA.B.3 5.NF.A 5.NF.B

Grade 5 students may be more likely to notice and apply patterns in their tables (using informal proportional reasoning). Some of them may try an approach like Strategy #2 in the Solutions where they explore the fraction relationships first and then try to find the difference of 2.5.

Grade 6 Common Core Standards: 6.RP.A 6.EE.B 6.EE.C

Sixth grade students should be encouraged to pay close attention to the ratios between the quantities. For example, they may discover that Marco's chocolate bar originally has 5/6 as much chocolate as Janine's. (For every 5 parts that Marco has, Janine has 6.)



Notice that the 'after' amount is 2/3 of Janine's and 4/5 of Marco's 'before' amount. Since Janine starts with 2.5 more ounces than Marco, each of the six parts of the diagram must stand for 2.5 ounces.

Grade 6 students with some algebra experience may attempt the Diving Deeper questions.

Solutions begin on page 39.

Name ________________________________

Problem 3

Try this mathematical experiment!

Step 1: Choose two whole numbers.
Step 2: Subtract the smaller number from the larger number.
Step 3: Subtract the difference and the subtrahend (larger – smaller).
Step 4: Repeat Step 3 until the answer is 0 or 1.

Find a way to predict which numbers will stop at 1.

Problem 3: *Testing the Waters*

There is no separate handout for Testing the Waters on this problem. This page includes a couple of suggestions for supporting students in solving the original problem.

- Keep the numbers that you choose fairly small. This will make the subtraction is less messy, and the patterns easier to notice.
- Rather than picking number randomly, choose your numbers in a predictable way.
- Organize your results in one or more tables.

Name ______________________________

Problem 3: *Diving Deeper*

Choose one or more questions to answer, or think of your own new questions to explore.

1. What happens if you keep going after you get an answer of 1?
2. What happens if you keep going after you get an answer of 0?
3. Will every choice of whole numbers lead to an answer of 0 or 1? How do you know?
4. When the answer is 0, you always subtract a number from itself at the end. How can you predict this number in advance? What causes this pattern?
5. Does it make sense to start with fractions, mixed numbers, or decimals? Explore.
6. Do some research on the *Euclidean Algorithm*. What is its purpose? Compare and contrast the *Euclidean Algorithm* to the process in this problem.

Problem 3
Teacher's Guide

Topics

Subtraction; factors and multiples, patterns

Materials

None

The grade level notes below are a rough guide for working with different age groups. Read through all three grades to get a feel for the likely range of students' thinking. Their approaches may vary a lot, so never feel limited to a particular grade level. Observe your students' ideas and strategies to assess their thinking, and build from there.

Grade 4 Common Core Standards: 4.OA.B.4 4.OA.C.5 4.NBT.B.4

Check that students know the vocabulary: minuend – subtrahend = difference.

Some students may worry that they don't know what numbers to try. Encourage them to make their own choices. At first, they may choose almost at random, but as they explore and observe, they should begin to have reasons for some of their choices. For example, they may investigate what happens when their numbers differ by 1, 2, or some other number. They may try large numbers in order to see if they will still go down to 1 or 0 eventually. Or they may make predictions about patterns they see and choose numbers to test their predictions.

Grade 5 Common Core Standards: ---

Fifth graders may make a little more progress, because they may be more fluent with subtraction and have an easier time recognizing multiples. Some of them may even notice a connection to greatest common factors.

Grade 6 Common Core Standards: 6.NS.B.4

Many students in grade 6 may be able try the Diving Deeper questions and to give reasons for what is happening in the problem. For example, they may recognize that

a multiple of n minus a multiple of n is always equal to another multiple of n, and
a multiple of n minus a non-multiple of n is never equal to a multiple of n.

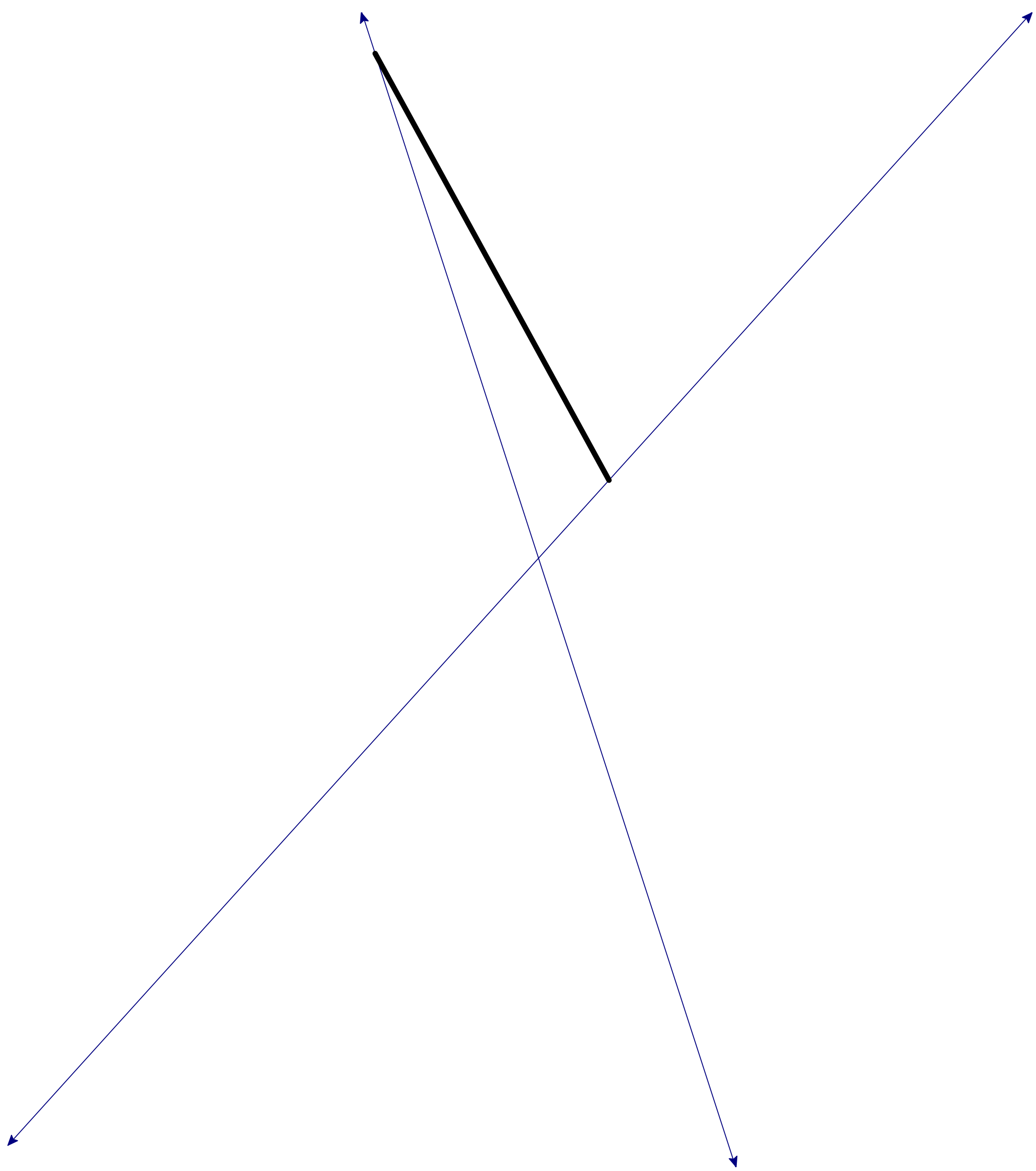
They may also use ideas like these to explain what causes the patterns in the problem.

Solutions begin on page 41.

Name ________________________________

Problem 4

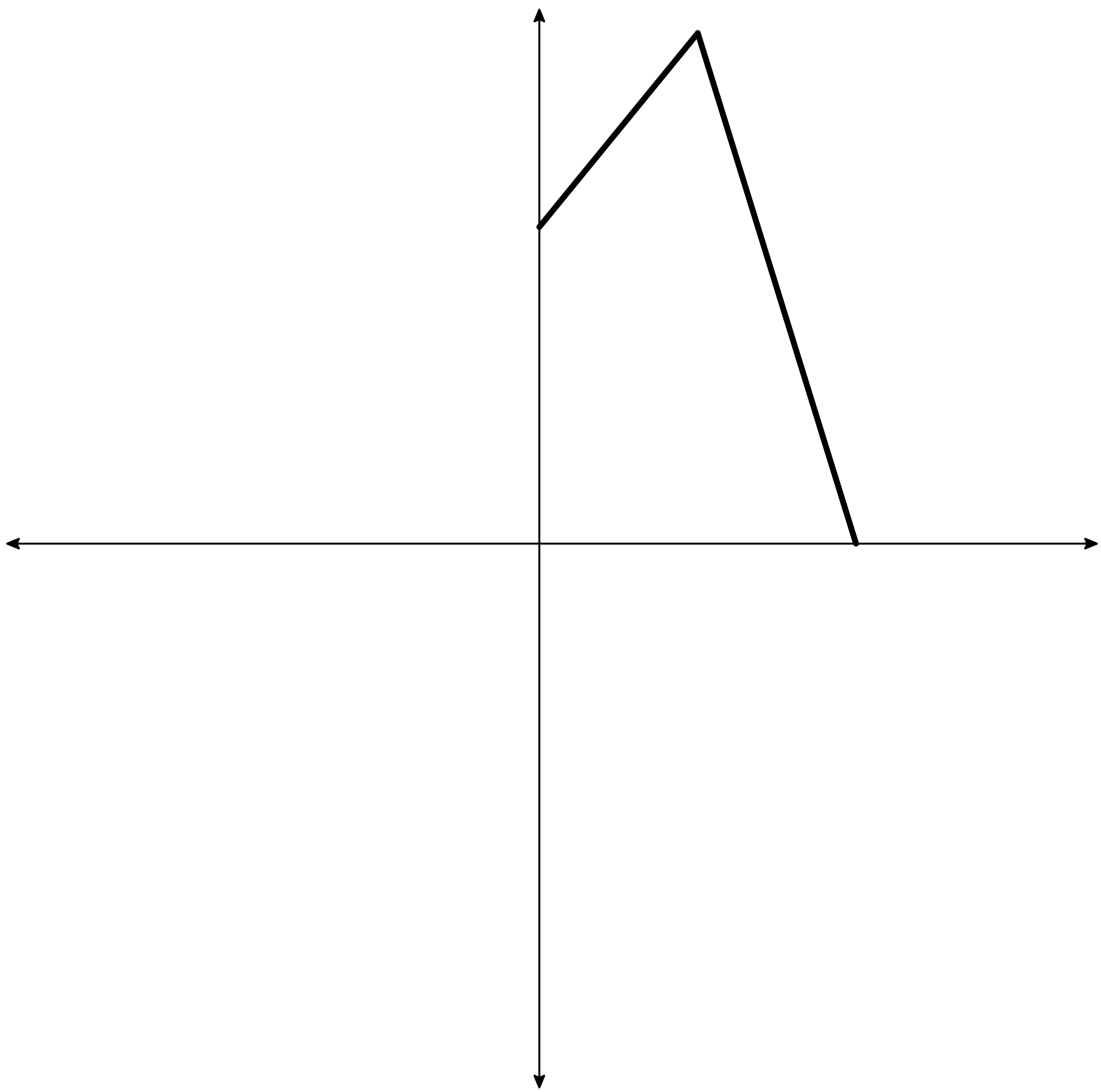
The dark segment is one side of a polygon. Draw the other sides so that both of the light lines are lines of symmetry for the polygon. Does the polygon have any other lines of symmetry?



Name ______________________________

Problem 4: *Testing the Waters*

The dark segments are two sides of a polygon. Draw the other sides so that both of the light lines are lines of symmetry for the polygon. Does the polygon have any other lines of symmetry?



Name ______________________________

Problem 4: *Diving Deeper*

Choose one or more questions to answer, or think of your own new questions to explore.

1. Does your polygon from Problem 4 have any other kinds of symmetry?
2. Watch how it affects the entire polygon when you change the original segment. Can you make polygons that have more or fewer than six sides just by making small changes in this segment?
3. What is the angle between the lines of symmetry? What happens if you change it? Experiment!

Problem 4
Teacher's Guide

Topics

Reflections; symmetry; length and angle measurement

Materials

Mira® math reflection tool (if available); ruler; compass; protractor

Notes: Make these materials available to students, but let them decide if and how to use them. Students may need more than one copy of the handout to experiment with.

The grade level notes below are a rough guide for working with different age groups. Read through all three grades to get a feel for the likely range of students' thinking. Their approaches may vary a lot, so never feel limited to a particular grade level. Observe your students' ideas and strategies to assess their thinking, and build from there.

Grade 4 Common Core Standards: 4.G.A.3 4.MD.C.5

Most grade 4 students will focus on visualizing the reflections and making connections between symmetry and reflections. A math reflection tool like Mira® can help them imagine and draw the other sides of the polygon. Alternatively, folding the paper helps them picture where the reflected sides will land. If they have trouble getting started, give them the Testing the Waters problem instead. Reasonable first goals are to create a rough sketch of the polygon (good enough to observe the reflection symmetries). If they are ready to

continue, ask them how they could make their drawings more accurate (without the Mira® tool). This may guide them to pay attention to the measurements of angles and lengths in their drawings. Remind them that they have rulers, protractors, and compasses available for measuring and drawing, and let them give it a try.

Grade 5 Common Core Standards: ---

Begin with the same process as for grade 4. Encourage fifth grade students to measure lengths and angles and use them to draw accurate reflections. Ask them to describe their observations and strategies.

Grade 6 Common Core Standards: --- (addresses some grade 8 standards)

Follow the suggestions for grades 4 and 5. Have them analyze their drawings and verbalize as many properties of reflections as they can. (In particular, ask them what happens if they join mirror points with a line segment. See Properties of Reflections in the Solutions.) For those who are prepared to continue further, the Diving Deeper questions offer plenty more to think about and explore. Note that some of this work extends the Common Core standards beyond the grade 6 level.

Solutions begin on page 43.

Name ______________________________

Problem 5

Is 110 feet per second a realistic highway speed limit? Explain.

Name ______________________________

Problem 5: *Testing the Waters*

How fast do you walk in feet per second and in miles per hour?
How do you know that both of your answers represent the same speed?

Name ______________________________

Problem 5: *Diving Deeper*

Choose one or more questions to answer, or think of your own new questions to explore.

1. Invent easier or faster ways to calculate or estimate the number of miles per hour for any number of feet per second. Explain your thinking.
2. Do the same for the reverse conversion: from miles per hour to feet per second.
3. Describe and explain connections and patterns between the reverse conversions.

Problem 5
Teacher's Guide

Topics

Multiplication and division of multi-digit numbers; converting between units; estimation

Materials

None

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Grade 4 Common Core Standards: 4.MD.A.1 4.MD.A.2 4.NBT.B.5 4.NBT.B.6

Many grade 4 students may find it helpful to begin with the Testing the Waters problem (especially early in the school year) about walking speeds. It involves smaller numbers and is easier to solve without long division. On the other hand, the original problem is approachable by students who are not yet fluent with multi-digit multiplication and division—if you give them plenty of time and encouragement to develop their own creative calculation methods (repeated addition and subtraction, grouping strategies, place value strategies, etc.). If there is time, ask students to share, compare, and justify their strategies.

Grade 5 Common Core Standards: 5.NBT.B 5.MD.A

Students in grade 5 may begin this problem without needing to try the Testing the Waters (TTW) problem first. However, TTW may help them develop common sense strategies that would not have occurred to them if they had jumped right into the original problem. It may even change their approach to the highway speed limit calculations. In either case, fifth graders will probably solve the problem more quickly than fourth graders. They should spend more of their time comparing and justifying strategies.

Grade 6 Common Core Standards: 6.NS.B

The original problem may not be particularly challenging for advanced learners in grade 6. Students who finish quickly should spend most of their time on the Diving Deeper questions, perhaps searching for multiple shortcuts and discussing advantages and disadvantages of each. (Note: Students who start with the simpler numbers in the Testing the Waters question may find it easier to discover patterns that lead to shortcuts.)

Solutions begin on page 45.

Name ______________________________

Problem 6

In	1	2	3	4	5
Out	5.0	5.5	6.0	6.5	7.0

Part 1: Invent a story or real-world situation for the table.
Part 2: Find a rule for the table.
Part 3: Predict the input when the output is 11.25.
Part 4: What do this input and output mean in your story?

Name ______________________________

Problem 6: *Testing the Waters*

In	1	2	3	4	5
Out	15	25	35	45	55

Part 1: Invent a story or real-world situation for the table.
Part 2: Find a rule for the table.
Part 3: Predict the input when the output is 235.
Part 4: What do this input and output mean in your story? Are they realistic?

Name ______________________________

Problem 6: *Diving Deeper*

Choose one or more questions to answer, or think of your own new questions to explore.

1. Graph the relationship between the input and output.
2. Write two or more rules for the table using variables.
3. Write equations that turn the output into the input. What do the rules tell you about your story? How do your original equations relate to your 'reversed' equations?
4. What is the rate of change of the relationship? Does the rate of change show up in your formulas? Why or why not?
5. Make your own table that has decimals in *both* the input and the output. Make a story for your table, and try to find a rule.

Problem 6
Teacher's Guide

Topics

Patterns and rules; adding decimals; linear relationships; rate of change

Materials

None

The grade level notes below are a rough guide for working with different age groups. Read through all three grades to get a feel for the likely range of students' thinking. Their approaches may vary a lot, so never feel limited to a particular grade level. Observe your students' ideas and strategies to assess their thinking, and build from there.

Grade 4 Common Core Standards: 4.OA.C.5 4.NF.C.7

Once students know what their stories will be about, the next step is to decide what the input and output stand for—and to be precise about their meanings. The best stories will be written so that the numbers are realistic (although it is also okay for students to write creative stories about imaginary situations, as long as the stories clearly fit the table).

The rules for the table may be described verbally or with algebraic expressions. If students have trouble taking half of a decimal, suggest they (1) draw a picture that shows the decimal's meaning in their story (or some other diagram like a number line) and find half of it visually, or (2) think of money.

Grade 5 Common Core Standards: 5.OA.A 5.OA.B 5.NBT.B.7 5.G.A

Fifth grade students may progress further. Some of them may move on to the Diving Deeper questions.

Grade 6 Common Core Standards: 6.NS.B.3 6.EE.A 6.EE.C

Some grade 6 students may spend much of their time on the Diving Deeper questions.

Solutions begin on page 47.

Name ________________________________

Problem 7

Aalok, Hector, Louisa, and Martin are collecting pennies.
Aalok has 4 times as many pennies as Hector.
Louisa has 2 times as many pennies as Martin.
Aalok has 9 times as many pennies as Louisa.

Part 1: Hector has _____ times as many pennies as Martin.
Part 2: Change 4, 9, and 2 to 40, 90, and 20. Answer the question again.
Part 3: What happens if you keep multiplying each number by 10?

Name ________________________________

Problem 7: *Testing the Waters*

Aalok, Martin, and Louisa are collecting pennies. Aalok has 6 times as many pennies as Martin. Louisa has 4 as many pennies as Martin.

Aalok has _____ times as many pennies as Louisa.

Name ______________________________

Problem 7: *Diving Deeper*

Choose one or more questions to answer, or think of your own new questions to explore.

Answer more questions about (the original) Problem 7.

1. Martin has ____ times as many pennies as Hector.
2. How many pennies can Martin have? Why?
3. Use variables to write an equation for each relationship. Use these equations prove that the answer to the original question is *always* 4.5.
4. Replace 4, 2, and 9 by a, b, and c. What is the answer now?
5. Prove that multiplying all three numbers in the original problem by 10 always makes the answer 10 times greater. (Question 4 may help!)

Problem 7
Teacher's Guide

Topics

Multiplicative comparisons (ratios); multiples; place value; algebraic expressions

Materials

Graph paper for drawing diagrams (optional)

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Grade 4 Common Core Standards: 4.OA.A 4.NBT.A.1

Most students will choose a number for one person and then figure out what the other people must have. (See Strategy #1 in the Solutions.) They will discover that certain numbers don't work, because some people will have a fractional number of pennies. Once fourth graders find something that does work, they may still find it challenging to compare Hector's and Martin's numbers, because neither is a multiple of the other.

Once students have found an answer, ask them to continue testing different numbers to make sure that the answer is always the same. If they are ready to explore further, see the comments for grade 5.

Grade 5 Common Core Standards: 5.NBT.A 5.NF.B

Grade 5 students are likely to take the same approach as fourth graders—choosing numbers and testing them. They may work more quickly because they are more familiar with multiples. A good challenge for fifth graders is to draw a bar diagram or a number line that shows the relationship between all of the quantities and illustrates why Hector must *always* have 4.5 times as many pennies as Martin, regardless of the total number of pennies. See Strategy #2 in the Solutions.

Grade 6 Common Core Standards: 6.RP.A 6.EE.A 6.EE.B

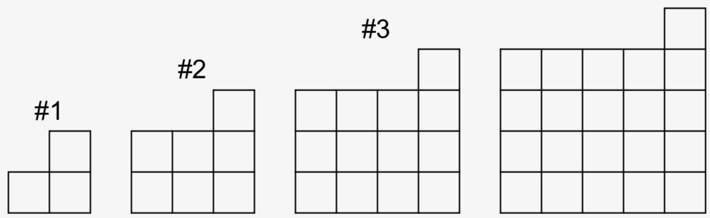
Grade 6 students will also probably begin by choosing numbers. This is good way to start, but afterwards, encourage them to look for more abstract methods that will prove that the answer is *always* 4.5. For example, they could draw a diagram (Strategy #2 in the Solutions) and, if possible, write algebraic equations that show the relationships between the variables. See the Diving Deeper problems and Strategy #3 in the Solutions.

Solutions begin on page 49.

Name ________________________________

Problem 8

Part 1: The perimeter of #1 is 8 units. What is the perimeter of #50?

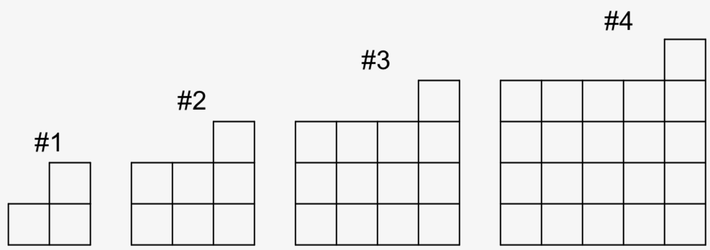


Part 2: Which shape number has a perimeter of 1000 units? 10,000 units?
Part 3: Make an in-out table and find a rule for it. (Find more than one rule if you can!)
IN: shape's number OUT: shape's perimeter

Name ________________________________

Problem 8: *Testing the Waters*

Part 1: Draw the next three shapes on graph paper.



Part 2: The perimeter of #1 is 8 units. What are the perimeters of #2 through #7?
Part 3: Describe any patterns you see in your answers.
Part 4: Predict the perimeter of #10 without drawing it. Explain your thinking.

Name ______________________________

Problem 8: *Diving Deeper*

Choose one or more questions to answer, or think of your own new questions to explore.

1. Figure out why the rules work.
2. Change the word *perimeter* to *area* everywhere that it appears in the problem. Change the numbers in Parts 1 and 2 as needed or desired. Then solve your new problem.
3. Make up your own problem using a shape pattern that you create. Trade problems with other students and solve them.

Problem 8
Teacher's Guide

Topics

Perimeter; patterns and rules; algebraic expressions

Materials

Graph paper for drawing shapes and making tables (optional)

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Grade 4 Common Core Standards: 4.OA.A.5 4.MD.A.3

If grade 4 students have trouble getting started, give them the Testing the Waters (TTW) problem first. TTW is basically the first part of Problem 8 with more scaffolding. Students begin by drawing more of the pictures, listing the perimeters, looking for patterns, and finding the perimeter of shape #10 instead of shape #50. Once they have finished all of this, some of them may be ready to return to the original problem.

Grade 5 Common Core Standards: 5.OA.A 5.OA.B

Some grade 5 students may find it helpful to begin with Testing the Waters (see the grade 4 notes), but more of them may be able to jump right into the original problem. See how much progress they can make on their own before providing extra scaffolding.

Grade 6 Common Core Standards: 6.EE.A 6.EE.C

Some grade 6 students may spend most of their time describing the patterns with algebraic expressions and explaining why the different expressions always give the same answer. Many of them will figure out the algebraic rules by looking at their table. However, they should also try to connect the expressions to the shapes themselves in order to explain why they work. (See Diving Deeper question 1 and "Why the rules work" in the Solutions for possible explanations.)

Solutions begin on page 51.

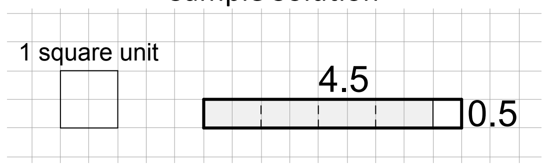
Intrepid Math Solutions

Problem 1

Part 1: 3 by 2 rectangle

Part 2:

Sample Solution



Interpreting the drawing

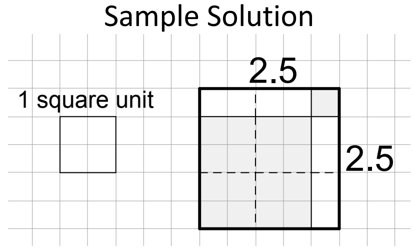
The shaded part holds 4 half-square-centimeters, or 2 square cm. (4 • 0.5)
The unshaded part holds ½ of a half-square-centimeter, or 0.25 square cm. (0.5 • 0.5)
The total area is 2 + 0.25 = 2.25 square cm.

Summary (distributive property)

4.5 • 0.5 = (4 + 0.5) • 0.5 = (4 • 0.5) + (0.5 • 0.5) = 2 + 0.25 = 2.25

Part 3:

Sample Solution



Interpreting the drawing

Lower-left: 2 groups of two square centimeters, or 4 square cm (2 • 2)
Lower-right: 2 half-square-centimeters, or 1 square cm (2 • 0.5)
Upper-left: 2 half-square-centimeters, or 1 square cm (2 • 0.5)
Upper-right: ½ of a half-square-centimeter, or 0.25 square cm. (0.5 • 0.5)
The total area is 4 + 1 + 1 + 0.25 = 6.25 square cm.

Summary (distributive property)

2.5 • 2.5 = (2 • 2) + (2 • 0.5) + (2 • 0.5) + (0.5 • 0.5) = 4 + 1 + 1 + 0.25 = 6.25

Notes

The 2 half-square-centimeters (2 • 0.5) can also be seen as ½ of 2 square cm (0.5 • 2).
Some students may want to explore the possibility of non-rectangular solutions.

Problem 1: *Testing the Waters*

Sample solutions

1 cm by 9 cm rectangle (9 sq cm)
2 cm by 8 cm rectangle (16 sq cm)
3 cm by 7 cm rectangle (21 sq cm)
4 cm by 6 cm rectangle (24 sq cm)
5 cm by 5 cm rectangle (25 sq cm)
11.5 by 0.5 cm rectangle (5.75 sq cm)

Patterns: There are many, including: One dimension increases by 1, while the other decreases by 1. The dimensions always have a sum of 10. The areas increase more and more slowly (by consecutive odd numbers: +7, +5, +3, +1) as the rectangles become more square-like.

Notes:

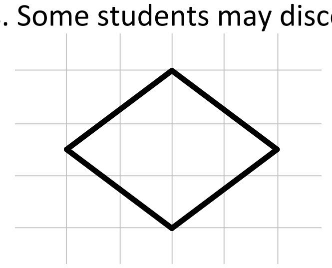
Students may believe that there are only five answers. The sixth shape may lead them to consider non-rectangular shapes or the possibility of working with decimals or fractions.

Problem 1: *Diving Deeper*

1. A rectangle with length 4.791 cm and width 0.209 cm is very close. Most students will estimate/test/revise. The area will never be exactly 1 cm^2 no matter how many decimal places they use. (The exact answer involves square roots and may be found with the quadratic formula.)

2. For shapes with a perimeter (or circumference) of 10 cm:
   The square (area 6.25 cm^2) has the greatest possible area of any rectangle.
   A circle has (area about 7.96 cm^2) has the greatest possible area of any shape.
   Any regular polygon with 6 or more sides has an area greater than 7 cm^2.

3. The following rhombus works. Some students may discover other solutions.



Notes: Most students will need to estimate measurements for Parts 2 and 3, though some may discover the circle in Part 2. Students who know the Pythagorean theorem can prove that the hexagon in Part 2 and the rhombus in Part 3 both work. (The hexagon is trickier to prove.)

Problem 2

Janine started with 15 ounces, and Marco started with 12.5 ounces.
Each person has 10 ounces of chocolate.

Strategies

#1 Start with a table using the difference of 2.5 (from sentence 2). Amount are in ounces.

Janine	3	6	9	12	15
Marco	0.5	3.5	6.5	9.5	12.5

Take away 1/3 of Janine's and 1/5 of Marco's chocolate.

Janine	2	4	6	8	10
Marco	0.4	2.8	5.2	7.6	10

Some students may save time by using patterns. For example, the difference between Janine and Marco starts at 1.6 decreases by 0.4 each time.

#2 Start with a table using the relationships from sentence 1. Amounts are in ounces.

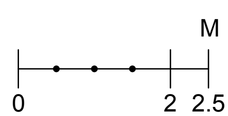
Janine	3	6	9	12	15
Marco	2.5	5	7.5	10	12.5

In the last column, notice that Janine's number is 2.5 greater than Marco's.
Check: 2/3 of 15 and 4/5 of 12.5 both equal 10.

Sample calculation for the table (first column)
Note: 2/3 of Janine's bar and 4/5 of Marco's bar are left.
Choose 3 ounces for Janine.
2/3 of 3 ounces equals 2 ounces.
2 ounces is 4/5 of Marco's amount (See the Suggestion below)
Marco's amount is 2.5 ounces.

Suggestion

Draw pictures or diagrams to visualize fraction relationships. For example:
2 is 4/5 of what number?

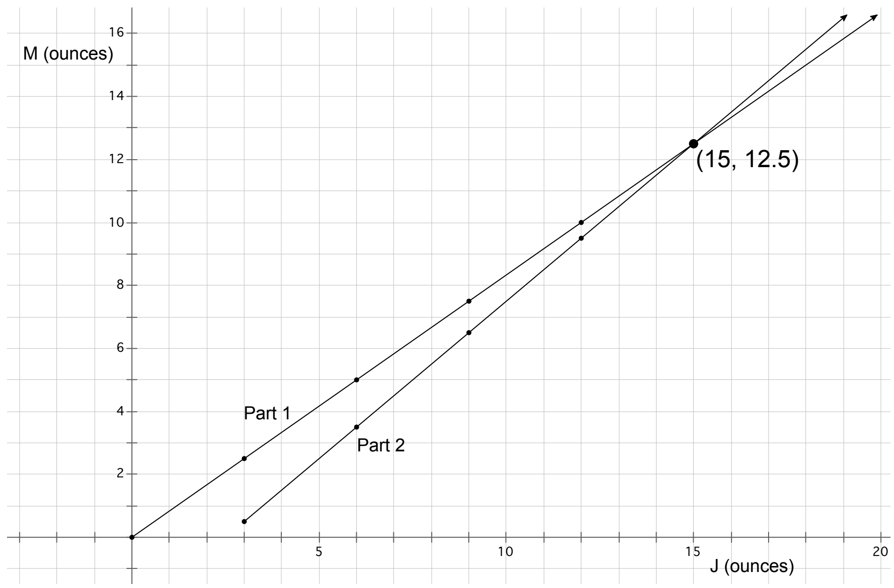


Problem 2: *Testing the Waters*

Each person has 4 ounces of chocolate.
Janine started with 12 ounces, and Marco started with 8 ounces.

Problem 2: *Diving Deeper*

1 and 2.



3. The solution occurs at (15, 12.5) where the two graphs intersect.

4. Sample equations for Part 1 graph: $M = \frac{5}{6} \cdot J$ or $J = \frac{6}{5} \cdot M$ or $\frac{2}{3} \cdot J = \frac{4}{5} \cdot M$
   Sample equations for Part 2 graph: $M = J - 2.5$ or $J = M + 2.5$ or $J - M = 2.5$

5. Sample equation for *M*: $\frac{6}{5} \cdot M = M + 2.5$
   Sample equation for *J*: $\frac{5}{6} \cdot J = J - 2.5$

6. Solution to the *M* equation: 12.5 Solution to the *J* equation: 15

Problem 3

You stop at 1 when the numbers you choose are not both multiples of the same number. (Of course, this does not include 1, because all whole numbers are multiples of 1!)

In other words, you stop at 1 when the numbers' *greatest common factor* is 1.

Examples of experiments

#1 Choose 14 and 33.

33 – 14 = 19   19 – 14 = 5   14 – 5 = 9   9 – 5 = 4   5 – 4 = 1

Stop, because the answer is 1.

You stop at 1 because 14 and 33 are not multiples of the same whole number (except 1).

#2 Choose 14 and 35.

35 – 14 = 21   21 – 14 = 7   14 – 7 = 7   7 – 7 = 0

Stop, because the answer is 0.

You never hit 1, because both numbers are multiples of 7.

Notes

Some students may notice that the number that they subtract from itself to get 0 is equal to the *greatest common factor* of the two original numbers. See Diving Deeper question (4).

In fact, the process in this experiment is a shortcut for finding greatest common factors of large numbers (by replacing them with smaller numbers that have the same greatest common factor). Some students may like to do some research on the *Euclidean Algorithm* after they have finished the problem. See Diving Deeper question 6.

Problem 3: *Diving Deeper*

1. If you keep going after you get an answer of 1, you will eventually reach 0. Depending on the subtrahend, it may take a long time to get there, because the difference will begin decreasing by only 1 at each step.

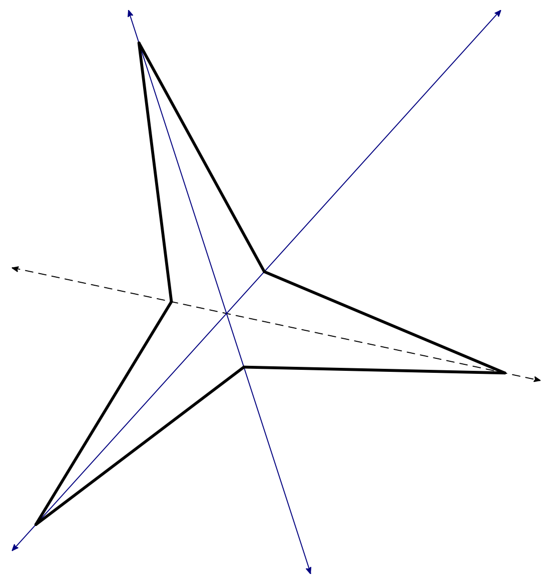
2. If you keep going after you get an answer of 0, the next difference will become the subtrahend and will remain that way forever.

3. Yes, every choice of whole numbers will lead to an answer of 0 or 1. In fact, every choice will eventually lead to 0, even if you reach 1 first. To understand why, think about (1) why the difference must always eventually become less than all (non-zero) differences that came before, and (2) why the difference can never become negative.

4. The number that you subtract from itself at the end is the greatest common factor of the pair of numbers that you began with. To understand why, think about what happens when you subtract two numbers that are multiples of the same number.

5. Play around with this, and see what you can discover!

Problem 4



The dotted line is a third line of symmetry.

A few observations and questions

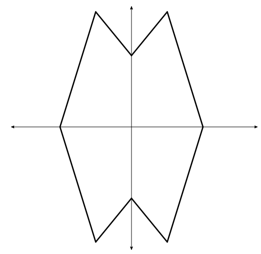
The lines of reflection are also the lines of symmetry. The original two lines of reflection form a 60° angle. The new line of symmetry makes 60° angles with these lines. The reflected segments join to make a closed figure in part because 60° • 6 = 360°. In one of the Diving Deeper questions, students explore what happens if they change this angle. Do the reflections still form a polygon? If so, what kind of polygon? If not, why not?

Some properties of reflections

(1) A reflected segment makes the same angle with the line of reflection that the original segment does.
(2) Reflected points are at equal distances from the line of reflection.
(3) If you join a point and its reflection with a line segment:
    - The segment is perpendicular to the line of reflection.
    - The line of reflection splits the segment into two congruent parts.

Note: The third property is close to the usual *definition* of a reflection.

Problem 4: *Testing the Waters*

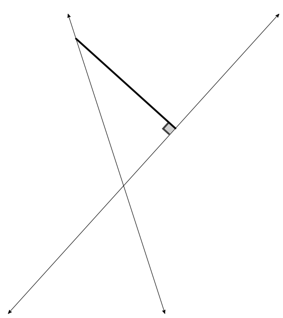


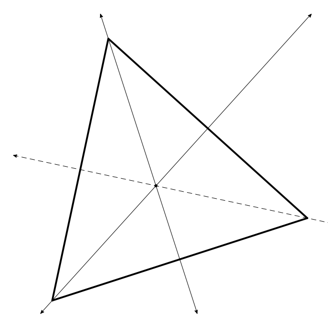
This polygon has no new lines of symmetry—only the two original ones. (It does have rotational symmetry: a half-turn around the point where the lines of reflection intersect.)

Problem 4: *Diving Deeper*

1. The polygon from the original Problem 4 also has rotational symmetry: a 1/3-turn (120°) around the point where the two lines of symmetry intersect).

2. The possibilities are endless. For example, by making the original segment perpendicular to a line of reflection, you can create an equilateral triangle.





   By 'bending' the original segment to create two or more connected segments, you can create polygons with more sides, etc. By creating symmetries between these segments, you can create additional symmetries in the final polygon, etc.

3. The angle between the lines of symmetry is 60°. This enables the segments to connect nicely when you reflect them, because 60 is a factor of 360. What happens when you choose other factors of 360? What happens when you choose angles with measures that are not factors of 360?

Problem 5

Yes. 110 feet per second is equal to 75 miles per hour.

Note

There are 5280 feet in 1 mile. Students may look this up or you may tell them.

Some calculation strategies

110 • 60 • 60 ÷ 5280	110 • 3600 ÷ 5280
110 ÷ 5280 • 60 • 60	110 ÷ 5280 • 3600
60 • 60 ÷ 5280 • 110	3600 ÷ 5280 • 110

Other strategies are possible. The top two strategies are probably the most common ones, but some students may think in unconventional ways. Ask them about the thinking that led to their strategies. Older or more advanced students may learn something from comparing and contrasting all six expressions and trying to explain why they give the same answer.

Problem 5: *Testing the Waters*

Sample solutions

2 miles per hour	about 2.9 or 3 feet per second
3 miles per hour	4.4 feet per second
4 miles per hour	about 5.9 or 6 feet per second

Getting started

Some students may estimate that they can walk (for example) a mile in 20 minutes and that this is 3 miles in an hour. Other students may begin with feet per second. For example, they may use a timer to count the number of seconds it takes them to walk across a room.

Converting the measurements

Long division may not be needed. For example:

2 miles per hour is 2 • 5280 = 10,560 feet per hour.
3 groups of 3600 is 10,800, which is a little bit greater than 10,560.
Therefore, 2 miles per hour is a little less than 3 feet per second.

Using this problem to find estimation shortcuts

By looking at numbers like the ones above, students may notice that the number of feet per second is a little less than 1.5 times greater than the miles per hour—or that the number of miles per hour is about 2/3 or 0.7 times the feet per second).

Problem 5: *Diving Deeper*

1. Some estimation shortcuts

fps ÷ 3 • 2 or fps • 2 ÷ 3	This one is a little less than the exact value.
fps ÷ 10 • 7 or fps • 7 ÷ 10	This one is a little greater than the exact value.

Some calculation shortcuts

fps ÷ 22 • 15 or fps • 15 ÷ 22	These are exact!
fps • $\frac{15}{22}$ or fps ÷ $\frac{22}{15}$	These are exact, too!
fps • 0.682 or fps ÷ 1.467	These are approximate. The decimals are rounded.

Notice that $\frac{3600}{5280}$ simplifies to $\frac{15}{22}$. Other shortcuts are possible.

3. For the reverse conversions, use the inverse operations. That is, interchange multiplication and division. For example, fps ÷ 3 • 2 becomes mph • 3 ÷ 2.

Problem 6

Part 1

Sample story

I pay $4.50 to buy a candy dispenser. After that, it costs $0.50 for each refill.

Input – the number of times I fill my candy dispenser

Output – the total cost

Part 2

Some possible rules

Take half of the input (or multiply it by 0.5) and add 4.5.

Add 9 to the input and divide the answer by 2.

Why do these two rules always give the same answer?

Part 3

13.5

Part 4

13 refills plus another half of a refill costs $11.25. This includes the cost of buying the dispenser.

Problem 6: *Testing the Waters*

Part 1

Sample story:

You are holding a walk to get donations to help support a family with large medical bills. Each person that sponsors your walk agrees to give $5 for you to join the walk plus $10 for each mile that you walk. The input is the number of miles that you walk, and the output is the number of dollars that you earn from each sponsor.

Part 2

If students say "Put a '5' after the *IN* number," ask them to use operations. Examples:

Multiply *IN* by 10 and add 5.

Subtract 1 from *IN*, multiply by 10, and add 15.

Part 3

23

Part 4

You would need to walk 23 miles to earn $235 from this sponsor, which is probably not realistic, because it is so far to walk.

Problem 6: *Diving Deeper*

1. The graph is a straight line. Check that students' points match the IN/OUT pairs. Students may decide whether or not to connect the dots. In cases where 'IN' has to be a whole number in order for the story to make sense, it is generally best not to connect the dots.

2. Possible rules include $y = 0.5 \cdot x + 4.5$ and $y = (x + 9) / 2$.
   x represents the 'IN' number (input) and y represents the 'OUT' number (output).

3. Possible equations include $x = (y - 4.5) \cdot 2$ and $x = 2 \cdot y - 9$.
   Using the story from the original solutions page 18, these equations tell you how many refills you can get for a certain number of dollars. In the 'reverse' equations, you do the inverse operations in reverse order.

4. The rate of change of the original relationship is 0.5, because the cost increases by $0.50 per refill. The number 0.5 shows up as the number multiplied by the 'IN' variable, x, in the equation $y = 0.5 \cdot x + 4.5$. (It also appears indirectly in the equation $y = (x + 9) / 2$, because dividing by 2 has the same effect as multiplying by 0.5.)

Problem 7

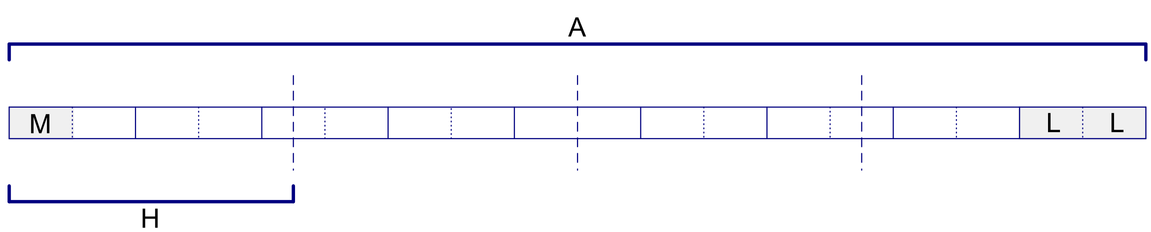
Part 1: Hector has 4.5 (4½) times as many pennies as Martin.
Part 2: Hector has 45 times as many pennies as Martin.
Part 3: The answer becomes 10 times greater each time you multiply all of the numbers by 10.

Strategies for Part 1

#1 Choose numbers.
Example: Aalok's number has to be a multiple of both 4 and 9. Choose 36.
Hector has 36 ÷ 4 = 9 pennies.
Louisa has 36 ÷ 9 = 4 pennies.
Martin has 4 ÷ 2 = 2 pennies.
9 is 4.5 times as many as 2. (2 • 4.5 is halfway between 2 • 4 and 2 • 5.)

Students should choose other numbers to check that the answer always comes out to 4.5.
It's not necessary to start with Aalok's number. Experiment with other possibilities.
When a number doesn't work, try another.
Strategies #2 and #3 show *why* the answer is always 4.5.

#2 Draw a diagram.
This picture illustrates the relationships between the quantities, which shows why the answer to Part 1 is *always* true.



#3 Use algebraic equations.
Write the relationships:

A = 4 • H L = 2 • M A = 9 • L

Therefore:

4 • H = 9 • L	(because they are both equal to A)
4 • H = 9 • (2 • M)	(because L = 2 • M)
4 • H = (9 • 2) • M	(associative property of multiplication)
4 • H = 18 • M	(calculation)
2 • H = 9 • M	(both sides are still equal if you take half of them)
H = 4.5 • M	(same reason)

The final two steps could be combined by taking one fourth of each side.

Problem 7: *Testing the Waters*

Aalok has 1.5 times as many pennies as Louisa.

Problem 7: *Diving Deeper*

1. Martin has $\frac{2}{9}$ times as many pennies as Hector. ($\frac{2}{9}$ is the reciprocal of 4.5.)

2. He must have an even number of pennies (so that Hector has a whole number.)

3. Note: This process also appears in Strategy #3 in the original Solutions on page 21.

   Write the relationships:

   A = 4 • H   L = 2 • M   A = 9 • L

   Therefore:

4 • H = 9 • L	(because they are both equal to A)
4 • H = 9 • (2 • M)	(because L = 2 • M)
4 • H = (9 • 2) • M	(associative property of multiplication)
4 • H = 18 • M	(calculation)
2 • H = 9 • M	(both sides are still equal if you take half of them)
H = 4.5 • M	(same reason)

4. *b* • *c* ÷ *a* (or *b* ÷ *a* • *c*, or *c* ÷ *a* • *b*)

5. Multiply each variable in Part 4 by 10.

   (10 • *b*) • (10 • *c*) ÷ (10 • *a*)

   Write this expression in the simpler form, 10 • (b • c ÷ a), or notice that multiplying by 10 twice and dividing by 10 once has the same effect as multiplying by 10 once.

   Note: Students who know standard algebra notation will leave out some of the multiplication symbols and write division as a fraction:

$$\frac{10b \cdot 10c}{10a} = \frac{10bc}{a} = 10\frac{bc}{a}$$

Problem 8

Part 1: 204 units
Part 2: Shape #249; Shape #2499
Part 3:

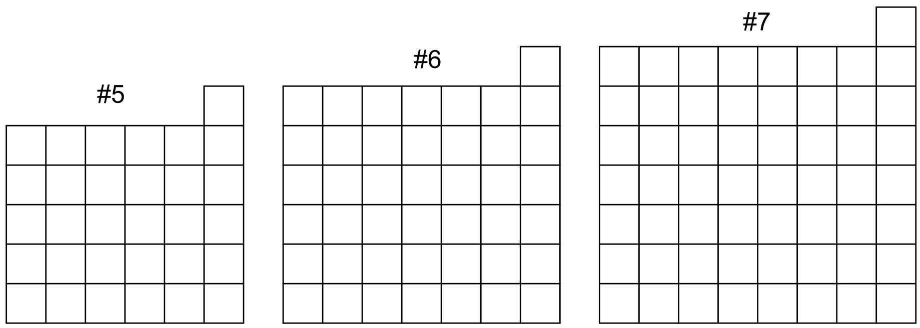
IN	1	2	3	4	5
OUT	8	12	16	20	24

Possible rules:
(1) Multiply IN by 4 and add 4. Or: OUT = IN • 4 + 4
(2) Add 1 to IN and multiply by 4. Or: OUT = (IN + 1) • 4

Note: Some students may discover that Parts 1 and 2 are easier if they figure out Part 3 first (but don't tell them)!

Problem 8: *Testing the Waters*

Part 1:



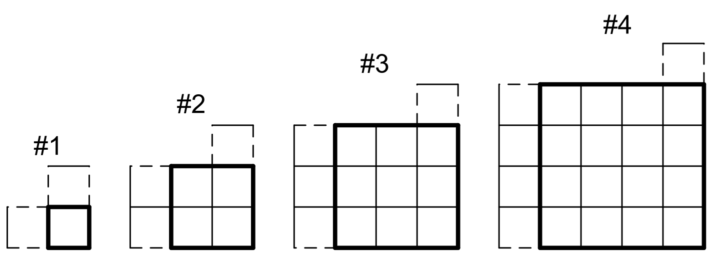
Part 2: 12, 16, 20, 24, 28, 32 (all cm)
Part 3: The pattern starts at 8 (for #1) and increases by 4 each time.
Part 4: The perimeter for #10 is 44 cm.
Strategy: Add 4 three more times (or add 12) to the perimeter of #7 (32).

Problem 8: *Diving Deeper*

1. Give the IN variable the name *N.* Sample explanations:

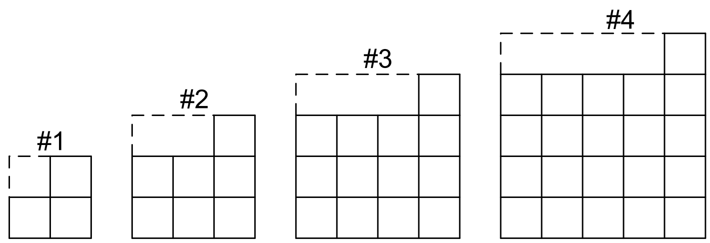
Rule 1 $N \bullet 4 + 4$

Shape #*N* has an *N* by *N* square inside of it, which has a perimeter of $N \bullet 4$.



When you join the top square and the left rectangle to the *N* by *N* square, the 4 dashed sides are added to the perimeter. (The other sides just replace the sides of the *N* by *N* square that get covered up.)

Rule 2 ($N + 1$) • 4



The perimeter of each shape is equal to the perimeter of an $N + 1$ by $N + 1$ square (Can you see why?), which is ($N + 1$) • 4.

2. If you assume that the area of shape #1 is 3 square units, then the pattern looks like

3, 7, 13, 21, 31, etc.

The numbers in the pattern increase by consecutive even numbers.

3 + **4** = 7   7 + **6** = 13   13 + **8** = 21   21 + **10** = 31

etc.

There are many possible formulas that students may discover, including

$N \bullet (N + 1) + 1$
$(N + 1)^2 - N$
$N^2 + N + 1$

Either the pattern or a formula may be used to make predictions for shapes farther down the list.