Deep Algebra Projects: Algebra 1 / Algebra 1
Step by Step

Topics

- Step functions
- The definition and rate of change of a linear function
- Connections between different representations of functions: words, tables, formulas, graphs, and real-world contexts
- Using formulas to stretch and compress graphs vertically and horizontally

The Step by Step project is about exploring step graphs. Students begin by figuring out how the *greatest integer* function works. They continue by drawing graphs and exploring how changes to the formulas affect the graphs. At the end, they apply what they have discovered in the following real-world context:

> *A parking lot allows you to park for free for any amount of time less than ½ hour. If you park for ½ hour or more, up to any amount less than 1 hour, you pay $3.00. You pay an additional $3.00 for each additional half hour.*

This story does not appear until late in the project, but you might consider introducing it right away so that students can be thinking about it as they work on the project. Specifically, they might imagine how a ticket machine for the ramp could be programmed with a formula that calculates the correct price when someone leaves.

There are a couple of optional handouts at the end of the project in case you would like to give students a coordinate grid that has been prepared in advance. Alternatively, you could ask students to figure out how to set up their own grids. I often give them a prepared grid early in the activity. Later, I have them create their own so that they gain the experience of setting up their scales in ways that make their graphs easy to draw and read.

Step by Step
Grade 8 Extension Project

Stage 1

In Stage 1, students learn how the *greatest integer* function works. They begin to explore how to represent this function using words, tables, formulas, and graphs. It may take some time for them to become comfortable with the "⟦ ⟧" symbol. The graphs may also be a little confusing at first, because students have probably not seen many examples of graphs that have discontinuous jumps, and they have almost certainly not thought about trying to represent graphs like these with formulas!

Handout 1 (at the end of the project) contains a prepared copy of the coordinate grid for Problems #2–4. If you choose to use it, a single copy per student will suffice during Stage 1.

What students should know:

- The definition of an integer
- How to locate positive and negative real numbers on a number line
- How to use an input/output table to create a graph

For Problem #4:

- The definition of a linear function
- Rate of change (slope) of a linear function

What students will learn:

- Know the meaning of the greatest integer operator, and calculate with it.
- Gain a deeper understanding of what a function's graph means.
- Begin to explore graphs of step functions.

Problem #1

$$[\![\pi]\!] = 3 \qquad [\![12.93]\!] = 12 \qquad [\![7]\!] = 7 \qquad [\![1\tfrac{5}{8}]\!] = 1$$

$$[\![-17.02]\!] = -18 \qquad [\![-13]\!] = -13 \qquad [\![-\sqrt{10}]\!] = -4 \qquad [\![-\tfrac{1}{2}]\!] = -1$$

"$[\![x]\!]$" is called the *greatest integer of x*.

Directions

- Describe a process for finding the *greatest integer* of a number.
- Explain your thinking using a variety of new examples like the ones above.

Diving Deeper

Write a precise definition of the *greatest integer of x*. Begin your sentence, "The *greatest integer of x* is …". Note: There is a subtle difference between stating what something *is* and explaining how to find it. State what it *is*. (You have already explained how to find it!) For example, do not say, "The *greatest integer of x* is what you get when you …".

Conversation Starters for Problem #1

What do you notice? What do you wonder?

I notice that the symbols remind me of absolute symbols, but they have “double bars.”

I notice that when the number inside the greatest integer symbol is an integer, the answer is the same as that number.

I notice that in the top row, you just eliminate the fraction or decimal part of the number.

I notice that this method does not work in the bottom row.

I wonder if the reason you can’t just eliminate the fraction or decimal in the bottom row is that the numbers are negative.

I notice that the answer is always less than or equal to the number inside the greatest integer symbol.

I wonder if enough examples were given to pin down the meaning of the *greatest integer* of a number?

I notice that the equations seem to involve a rounding process.

Solutions for #1

A process for finding the greatest integer of a number

To find the greatest integer of *x*, round *x* down to the nearest integer that is less than or equal to *x*. (The “or equal to” part of the statement means that if *x* is already an integer, the greatest integer of *x* is *x* itself.)

Note: In the language of computer programming, this is often called *floor* function.

Some additional examples

$[\![1.\overline{4}]\!] = 1$ $[\![29.6]\!] = 29$ $[\![138]\!] = 138$ $[\![12\tfrac{1}{3}]\!] = 12$

$[\![-17]\!] = -17$ $[\![-13.48]\!] = -14$ $[\![-\pi]\!] = -4$ $[\![-9\tfrac{2}{5}]\!] = -10$

Students should use their examples to support their description of the process. Encourage them to use a variety of values and notations for *x*: positive and negative numbers, fractions, decimals, irrational numbers, large and small numbers, etc.

Note

If your students know function notation, this is a good opportunity for them to use it. They could define $f(x) = [\![x]\!]$ and express equations like those above as $f(1.\overline{4}) = 1$, $f(-\pi) = -4$, etc. Throughout the activity, they may also replace *y* by *f(x)* in their tables and graphs.

The Diving Deeper question

The *greatest integer of x* is the greatest integer that is less than or equal to *x*.

Note: This definition helps to explain why the term *greatest* integer is used, even though the process involves rounding *down*—a situation that is often confusing for students. (This definition uses more precise language than “rounding down.”)

Problem #2

x	–2												2
$[\![x]\!]$													

Directions

- Choose more values for *x* and complete the table.
- Plot the ordered pairs from your table in a coordinate grid. (See Handout 1).
- Plot more points for inputs between –2 and 2.

Conversation Starters for Problem #2

What do you notice? What do you wonder?

I wonder if all of the inputs must be between –2 and 2.

It isn't absolutely necessary to keep all of the inputs between –2 and 2, but the table will be easier to read and understand if the inputs are listed in order.

I wonder if the inputs should increase at a constant rate in the table.

Again, the table may be a little easier to read and understand if the inputs increase at a constant rate, but it is not necessary to do this if it feels awkward. It is probably more important to choose inputs that help you to understand how the greatest integer function works.

I notice that all of the outputs in the table are integers (between –2 and 2).

I notice that there are no points on the graph between the horizontal grid lines.

I wonder what causes this.

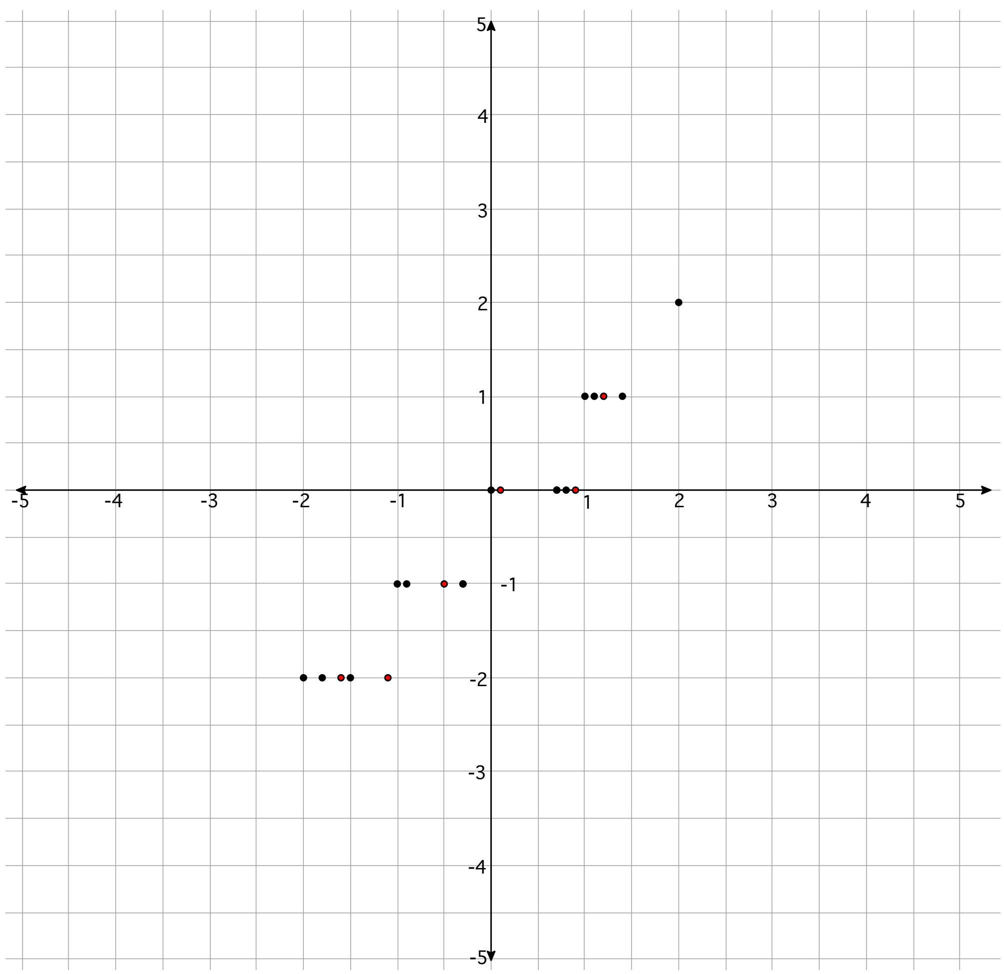
I notice that some parts of my graph look "flat."

Solutions for #2

Sample table

x	−2	−1.8	−1.5	−1	−0.9	−0.3	0	0.7	0.8	1	1.1	1.4	2
$[\![x]\!]$	−2	−2	−2	−1	−1	−1	0	0	0	1	1	1	2

Sample graph



Students' points will vary depending on their tables and the additional inputs they chose. However, all of their points should lie on the step graph shown in Problem #3 (and all should be between $x = -2$ and $x = 2$.) The black dots show points from the table above. The red dots show a few chosen additional points. You may or may not feel the need to have students distinguish between the two.

Problem #3

You have begun graphing the algebraic formula, $y = [\![x]\!]$.

Directions

- Complete your graph to show the outputs for all inputs between –5 and 5.
- Explain your thinking.
- Describe the shape of your graph, and use the shape to explain how the output changes as the input increases.

Conversation Starters for Problem #3

What do you notice? What do you wonder?

I wonder if I can just connect the points that I have plotted.

Be careful! Make sure that every point on your graph represents a correct input/output pair (and that every correct input/output pair between –5 and 5 is included in your graph).

I notice that it helps to keep plotting more points until I can predict the exact shape of the graph.

I notice that some of the flat parts of my graph need to continue to the left and/or right beyond the points that I plotted.

I notice that the title of this project gives a hint about the shape of the graph!

I wonder if I should connect the horizontal steps with vertical segments in order to connect all parts of the graph.

Although it is tempting to make the graph look connected, it is best not to—because it *isn't* connected! The outputs make sudden jumps (skipping a lot of numbers in between).

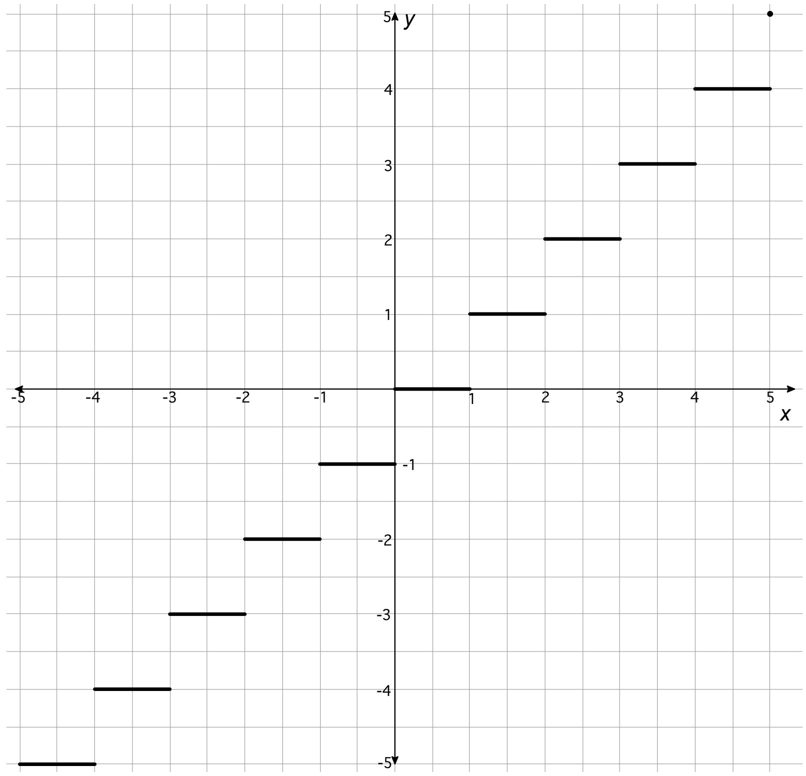
I notice that my graph does not make it clear what the output is when the input is an integer (because it shows two possible answers).

I wonder how to make this clear on my graph.

Try inventing your own method! (See Problem #4.)

Solutions for #3

The graph



Explanation

I saw that the points I plotted lined up horizontally and then jumped by one unit at each *x* integer. I realized that there could be no *y*-values between the integers, because all of the outputs are integers. This meant that I could not "connect dots" between the jumps.

Description of the graph

The graph looks like a series of identical, equally-spaced steps climbing from left to right. This means that as the input increases, the outputs stay constant for 1 unit and then suddenly jump to the next-higher integer value of *y*.

Problem #4

Although the graph of the greatest integer function is not a line, it does contain a linear pattern.

Directions

- Identify and describe a linear pattern within the greatest integer function. State its rate of change (slope). Explain your thinking.
- Explain why the greatest integer function is not linear, even though it contains this linear pattern.
- Your graph shows two outputs for certain inputs. Explain why this is a problem, and invent a method to fix your graph so that it shows which of the two outputs is correct in each case. Show your revised graph.

Conversation Starters for Problem #4

What do you notice? What do you wonder?

I notice that the left (or right) sides of the steps line up.

I notice that I could make a linear graph by connecting these points, but the "in-between" points do not belong to the original graph.

I notice that the rate of change stays the same for a while, but then the output makes a sudden jump.

I notice that whenever one input has two outputs, there are two points on the graph that line up vertically.

I notice that I can decide which output is correct by remembering that the greatest integer of an integer is equal to itself.

Solutions for #4

A linear pattern within the function

The outputs at each step are 0, 1, 2, 3, 4, 5, etc. This sequence forms a simple linear pattern that starts at 0 and has a constant rate of change of 1.

Why the function is not linear

The function is not linear because the graph is not a straight line. In particular, each output stays constant for a while before jumping to the next value. The output of a linear function would increase steadily, encountering all intermediate values of *y*.

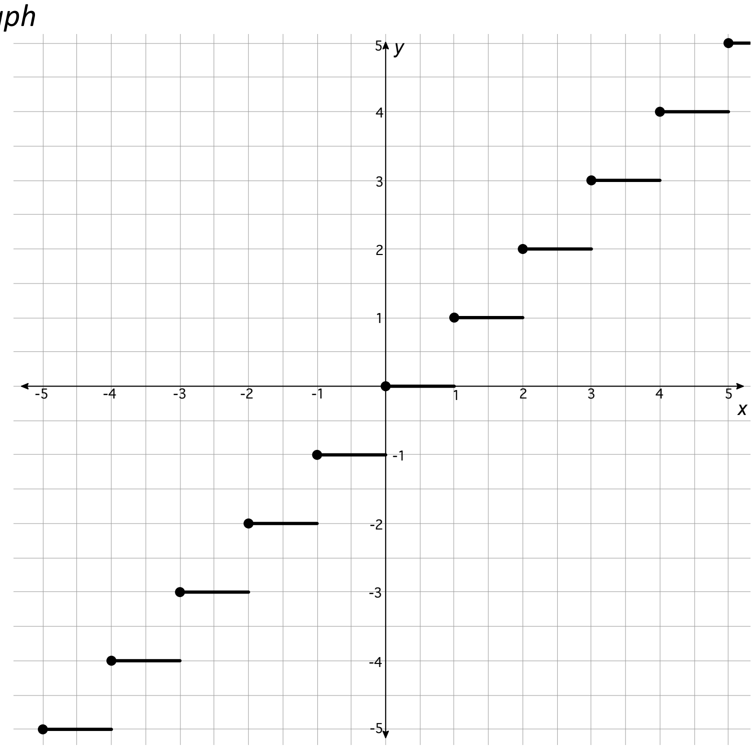
Some students may be more specific: the ratio of the change in *y* to the change in *x* is not the same for all pairs of points. For example, consider the following pairs of points:

Points: (1.2, 1) and (2.6, 2)	Ratio: (2 – 1) / (2.6 – 1.2) = 1 / 1.4 = 5/7
Points: (3.9, 3) and (3.5, 3)	Ratio (3 – 3) / (3.9 – 3.5) = 0 / 0.4 = 0

Ensuring that no inputs show more than one output on the graph

The graph appears to show two outputs at each integer value of *x*. One way to show which output is correct is to place a dot on the left side of each step to indicate that its *y*-value is the correct one for that integer input. For example, the dot on the left side of the *y* = 1 step shows that $[\![1]\!]$ is equal to 1 (not 0 as the step below it might suggest).

The revised graph



Stage 2

In Stage 2, students begin to explore how to make adjustment to their equations in order to produce desired effects on their graphs. If you plan to use Handout 1 for problems 5 and 6, each student will need a separate copy for each problem. No handout is available for Problem 7. Students are encouraged to set up their own coordinate system for this one.

What students should know:

- The ideas from Stage 1
- How to multiply a negative number by a positive number (or to use a calculator for this)
- How to translate flexibly between a formula, table, and graph

What students will learn:

- Understand formulas, tables, and graphs of simple step functions.
- Explore how it affects a graph when you multiply the input or output by a number.

Problem #5

Notice that the steps in your graph of $y = [\![x]\!]$ jump 1 unit each time.

Directions

- Change the formula so that the steps jump by only half of a unit each time.
- Show a new table and graph for your formula.
- Explain what causes the graph to change as it does.

Diving Deeper

- Find the rate of change of the linear pattern suggested by this new function. Explain. (Make a sequence from the natural number inputs and their outputs.)
- Sketch the graphs of $y = \sqrt{x}$ and $y = |x|$. Change these two formulas in the same way that you changed $y = [\![x]\!]$ above. Does the change affect these graphs in the same way as before? Explain.

Conversation Starters for Problem #5

What do you notice? What do you wonder?

I wonder how much of a change I will have to make to the original formula.

I notice that it makes sense to keep using the greatest integer symbol, because the graph is still a step graph.

I notice that it helps to make a revised table first—even though the directions ask for the formula first.

The table shows what has to happen to the outputs in order to make the jumps smaller.

I notice that it helps to draw the revised graph even before I make the table!

I notice that all of the outputs have to get divided by 2.

I wonder how to make this happen in my formula.

Solutions for #5

The new formula

To get the new formula, multiply the given greatest integer expression by $\frac{1}{2}$.

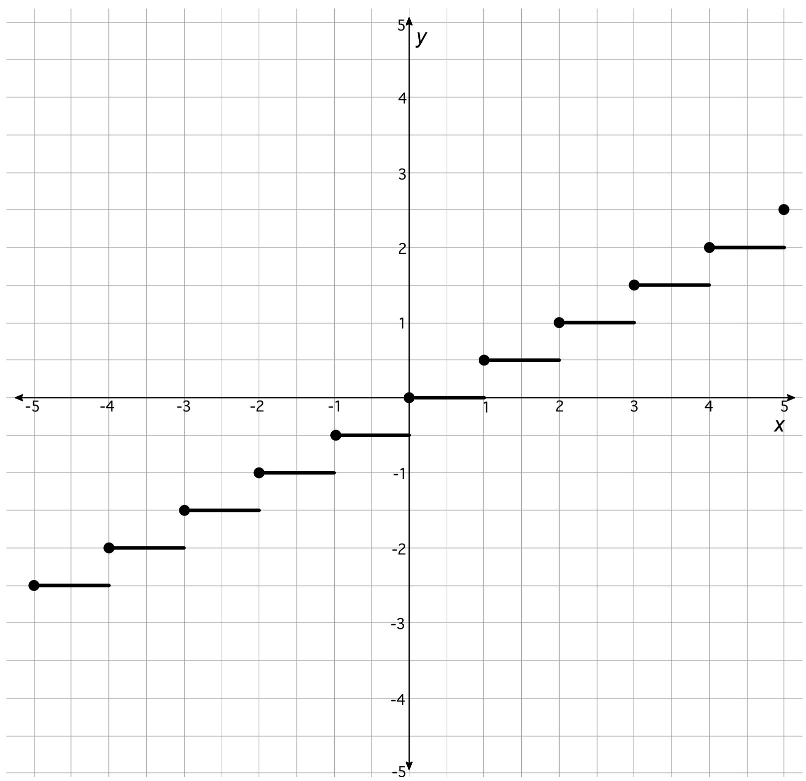
$$y = \frac{1}{2}[\![x]\!]$$

Note: This has the effect of making each output $\frac{1}{2}$ as large (half as far from 0).

A new table

x	−2	−1.8	−1.5	−1	−0.9	−0.3	0	0.7	0.8	1	1.1	1.4	2
$[\![x]\!]$	−1	−1	−1	−0.5	−0.5	−0.5	0	0	0	0.5	0.5	0.5	1

The new graph



What causes the change in the graph?

Because every *y*-value becomes half of what it was, the steps are squeezed vertically in toward the *x*-axis. (Each point is half as far away from the *x*-axis as it was in the original graph.)

Problem #6

Notice that the steps in your graph of $y = [\![x]\!]$ are 1 unit wide.

Directions

- Change the formula so that the steps are 2 units wide.
- Show a new table and graph for your formula.
- Explain what causes the graph to change as it does.

Diving Deeper

- Find the rate of change of the linear pattern suggested by this new function. Explain.
- Sketch the graphs of $y = \sqrt{x}$ and $y = |x|$. Change these two formulas in the same way as you did in Problem #6. Does the change affect these graphs in the same way as before? Explain.

Conversation Starters for Problem #6

What do you notice? What do you wonder?

I notice that it seems harder to adjust the width of the steps than to change to amount that each step jumps.

I notice that it helps to make a new graph and table before I try to create a new formula.

I notice that the best way to solve this problem is to experiment with making different changes to the formula and watching what happens to the original graph.

I notice that using addition or subtraction in my formula moves the entire graph but does not change the size of the steps.

I wonder if it matters whether I use multiplication or division in my formula.

I wonder if it matters whether I multiply numbers inside or outside the greatest integer symbol.

I notice that moving the multiplication inside or outside the greatest integer symbol changes the order in which I carry out the operations.

I notice that changing the order in which I carry out the operations usually changes the output.

I notice that the width of the steps changes in the opposite of the way that I expected.

I wonder why this happens.

I notice a connection with reciprocals.

> Although it is a difficult connection for most students to notice, some of them may discover that the width of the steps scales by the *reciprocal* of the number that they multiply the input by. Specifically, if they multiply the input by x, the step becomes $1/x$ times as wide. This works whether x is a whole number or not.

Solutions for #6

The new formula

To get the new formula, multiply the input (x) by $\frac{1}{2}$.

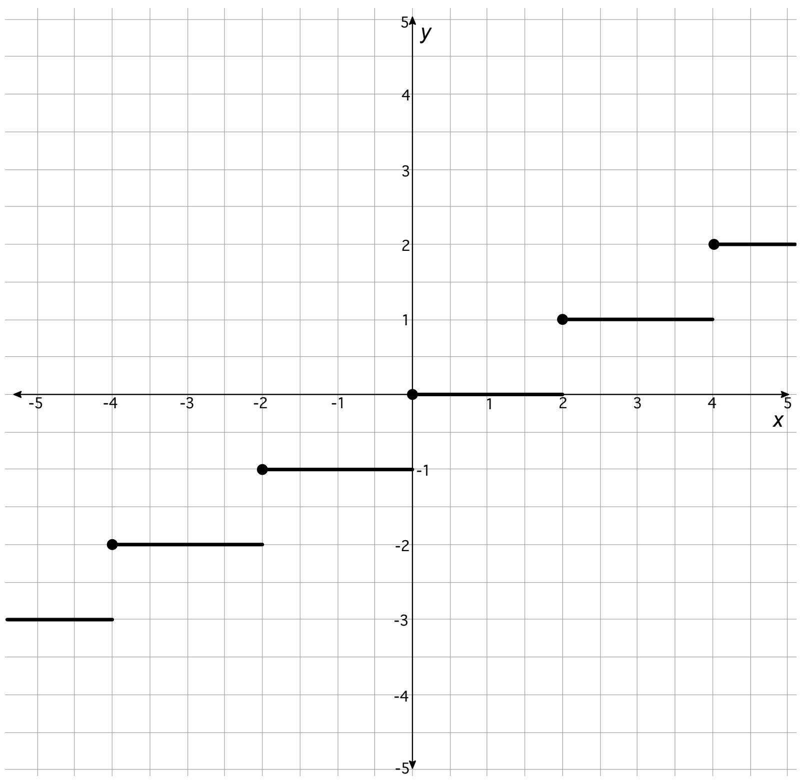
$$y = \left[\!\left[\frac{1}{2}x \right]\!\right]$$

Note: This has the—perhaps surprising—effect of making each input in the original table double (twice as far from 0 in the graph). Each input must be twice what it was in order to produce the same output!

A new table

x	−4	−3.6	−3	−2	−1.8	−0.6	0	1.4	1.6	2	2.2	2.8	4
$[\![x]\!]$	−2	−2	−2	−1	−1	−1	0	0	0	1	1	1	2

The new graph



The cause of the change in the graph

Because every x-value becomes twice as great, the steps are stretched horizontally away from the y-axis. (Each point is twice as far from the y-axis as it was in the original graph.)

Problem #7

You can combine ideas from the two previous problems.

Directions

- Create a formula for a step graph whose steps jump by 1 fifth of a unit each time and are 1 third of a unit wide.
- Show a table and graph for your formula.
- Explain your thinking for this problem.

Diving Deeper

- Find the rate of change of the linear pattern suggested by this new function. Explain.
- How would you change the formula to make the steps $\frac{2}{3}$ (or some other fraction between 0 and 1) as wide? What if you wanted to make them 1.8 times wider?

Conversation Starters for Problem #7

What do you notice? What do you wonder?

I wonder if it would be easier to begin with a table and graph before working on the formula.

I notice that it is convenient (though not necessary) to make the inputs increase by thirds in the table.

I notice that it may work better to make my own coordinate system for this graph so that I can adjust the scales.

I notice that it should be possible to make both changes at once, because one involves multiplying the input, and the other involves multiplying the output.

I notice that this problem is asking for a different kind of change than the previous problems did: the steps are getting narrower.

I notice that the steps get narrower when I multiply the input by larger numbers.

I wonder how to calculate the exact number that I need to multiply by x in order to make the step-measurements smaller.

Solutions for #7

The new formula

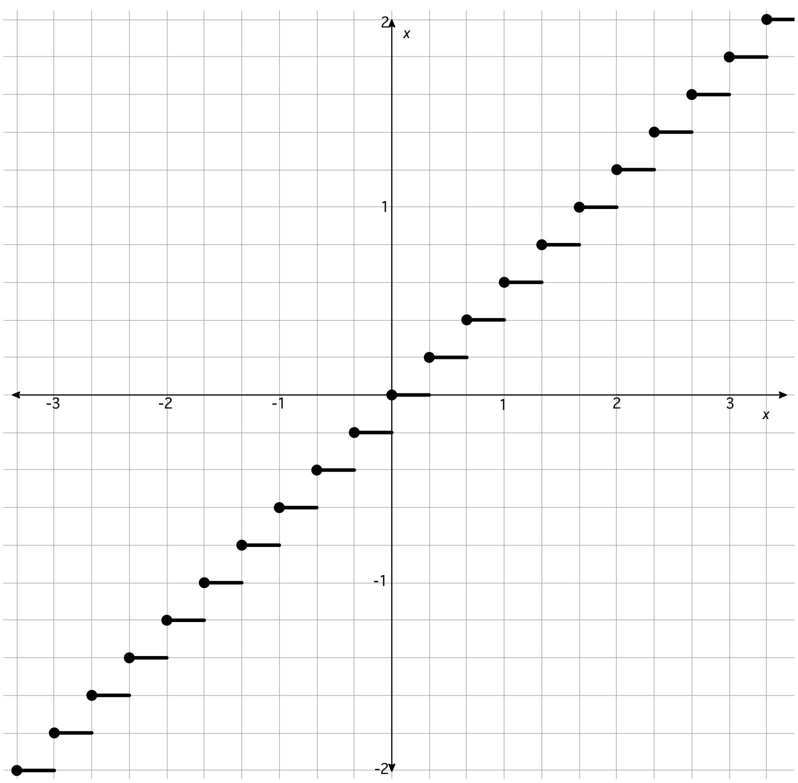
To get the new formula, multiply the input by 3 and the output (the entire expression) by $\frac{1}{5}$.

$$y = \frac{1}{5}[\![3x]\!]$$

A new table

x	−2	$-1\frac{2}{3}$	$-1\frac{1}{3}$	-1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$1\frac{1}{3}$	$1\frac{2}{3}$	2
$[\![x]\!]$	−1.2	−1	−0.8	−0.6	−0.4	−0.2	0	0.2	0.4	0.6	0.8	1	1.2

The new graph



Explanation

Multiplying *x* by 3 forces you to start with 1 third of the input in order to get the same output as in the original formula, which makes the steps 1 third as wide. Multiplying the entire expression by 1 fifth makes the output 1 fifth as great for each input, which means that each *y*-value becomes 1 fifth as great, causing the entire graph to be squeezed toward the *x*-axis.

Stage 3

In Stage 3, students apply their knowledge of the greatest integer function to represent a real-world situation.

In Stage 2, they should have noticed that there is a distinction between "multiplying inside or outside the greatest integer symbol." Encourage them to understand this distinction more mathematically—as the difference between multiplying a number by the input or by the output.

Handout 2 is available as an option for students to use for the graph in Problem #8.

Students who are attempting the second Diving Deeper question for Problem #9 may need some help interpreting the notation:

> $f\left(\llbracket x \rrbracket\right)$ involves taking the greatest integer of each input and then finding the output of the function using this input. $\llbracket f(x) \rrbracket$ involves finding the output of the function at each input and then taking the greatest integer of the result.

What students should know:

- The ideas from Stages 1 and 2

What students will learn:

- Apply discoveries about transformations of graphs in a real-world context.
- Explore more complex functions involving the greatest integer.

Problem #8

A parking lot allows you to park for free for any amount of time less than ½ hour. If you park for ½ hour or more, up to any amount less than 1 hour, you pay $3.00. You pay an additional $3.00 for each additional half hour.

Directions

- Explain why this situation is represented by a step graph.
- Create a table and graph for the situation. Label everything appropriately.
- Create a formula for the situation. Explain your thinking.

Diving Deeper

1. Find the rate of change of the linear pattern within the function $y = A[\![Bx]\!]$. Explain.
2. Create your own real-world scenario that uses a step graph. Find a formula, table, and graph for your scenario. Explain your thinking.
3. Change the wording of Problem #7 slightly:

   A parking lot allows you to park for free for any amount of time less than or equal to ½ hour. If you park for more than ½ hour up to any amount less than or equal to 1 hour, you pay $3.00. You pay an additional $3.00 for each additional half hour.

   - Explain or show how these small changes affect the graph and the table.
   - Adjust your formula to reflect the changes. Hint: This is very challenging! First, try to adjust the basic formula, $y = [\![x]\!]$ in a similar way. Then apply what you learn to your formula.

Conversation Starters for Problem #8

What do you notice? What do you wonder?

I notice that the price stays the same for a while and then jumps.

I notice that this situation calls for a formula containing the greatest integer symbol.

I notice that it makes sense to make time the input and cost the output.

I notice that my inputs and outputs will all be positive (unlike the earlier problems).

I notice that I may need to adjust the scales on my axes to fit the situation.

I notice that the adjustments needed to make my formula fit the situation are similar to the ones I learned about in earlier problems.

I wonder how to determine whether the dots are on the left or right end of each step.
Read the problem closely. The details of the wording contain the necessary clues.

I notice that I can test my formula by substituting different times for *x* and seeing if they give the correct cost every time.

I wonder if my formula could be used in the programming code for a parking ramp ticket machine to make it calculate the correct cost when someone leaves the ramp.

Solutions for #8

Why the situation is represented by a step graph

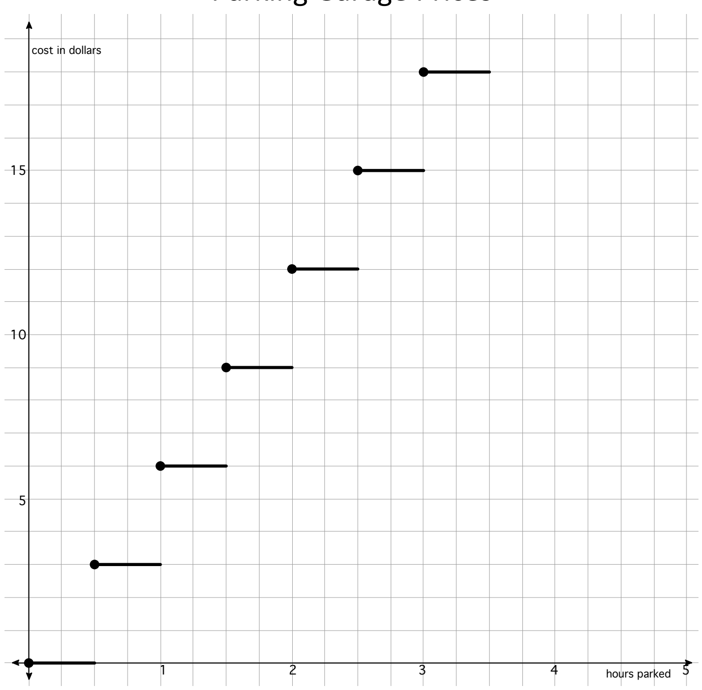
The parking lot charges are represented by a step graph because the price stays constant for a time and then jumps suddenly.

A table

x	0	0.3	0.5	0.7	1.0	1.1	1.4	1.5	1.7	1.8	2.0	2.3	2.5	2.9
y	0	0	3	3	6	6	6	9	9	9	12	12	15	15

The graph

Parking Garage Prices



The formula

In order to make the price jump every half-hour, the steps must be half a unit long. You accomplish this by multiplying *x* by 2. In order to make the price increase by $3.00 each half-hour, the steps must jump 3 units each time. You accomplish this by multiplying the entire expression by 3. Combining these two changes gives

$$y = 3[\![2x]\!],$$

where *x* represents the time in hours and *y* stands for the cost in dollars.

Problem #9

The greatest integer operator, $[\![\ ]\!]$, may be used in combination with other mathematical functions. Although their graphs may look more complex than the ones you have seen so far, it affects them in predictable ways.

Directions

- Graph the functions $y = [\![x]\!]^2$ and $y = [\![x^2]\!]$.
- Compare and contrast the two graphs.

Diving Deeper

- Explore graphs of other functions that contain the greatest integer operator.
- Discover and describe general methods for changing the graph of $y = f(x)$ into the graphs of $y = f([\![x]\!])$ and $y = [\![f(x)]\!]$.

Conversation Starters for Problem #9

What do you notice? What do you wonder?

I wonder if the two graphs will be the same.

I notice that it helps to begin by plotting a lot of points so that I can get a feel for how the equations work.

I notice that moving the exponent to the inside or outside of the greatest integer symbol changes the order of the calculations.

I notice that the two formulas do not always give the same output for the same input (which means that the two graphs will *not* be the same).

I wonder how the graphs will compare to the graph of $y = x^2$ (without the greatest integer symbol).

I notice that there are no negative outputs in either graph, which means that neither graph will drop below the *x*-axis.

I notice that the steps and jumps are sometimes but not always of equal length in these graphs.

I wonder how I can figure out the exact widths of the steps in the second graph.

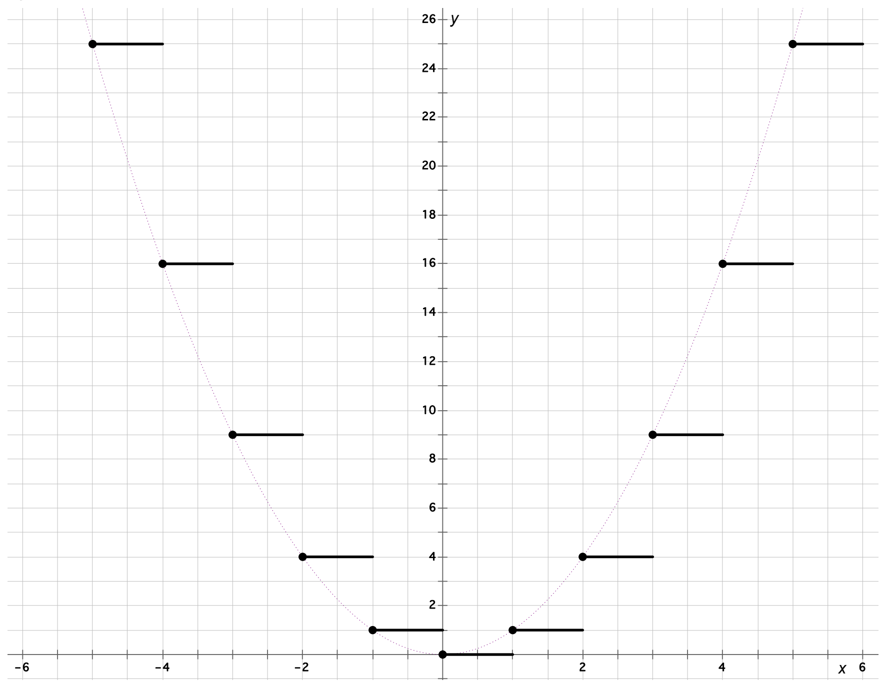
I notice that I can use square roots to help me draw the second graph.

I notice that the dots on my steps (for both graphs) land on the graph of $y = x^2$.

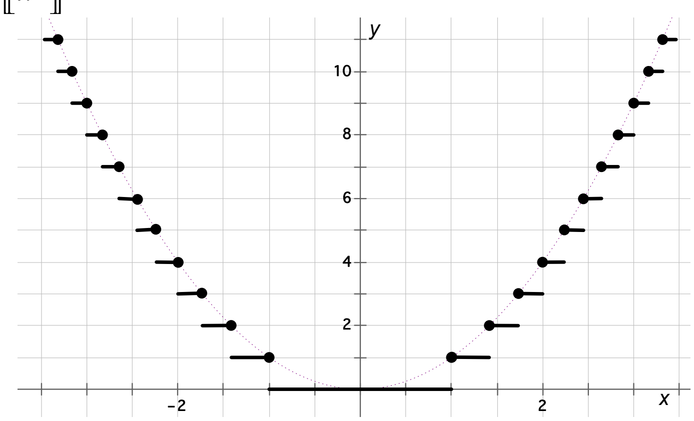
I notice that pieces of the graph of $y = x^2$ look like they get "flattened out" in both graphs.

Solutions for #9

The graph of $y = [\![x]\!]^2$



The graph of $y = [\![x^2]\!]$

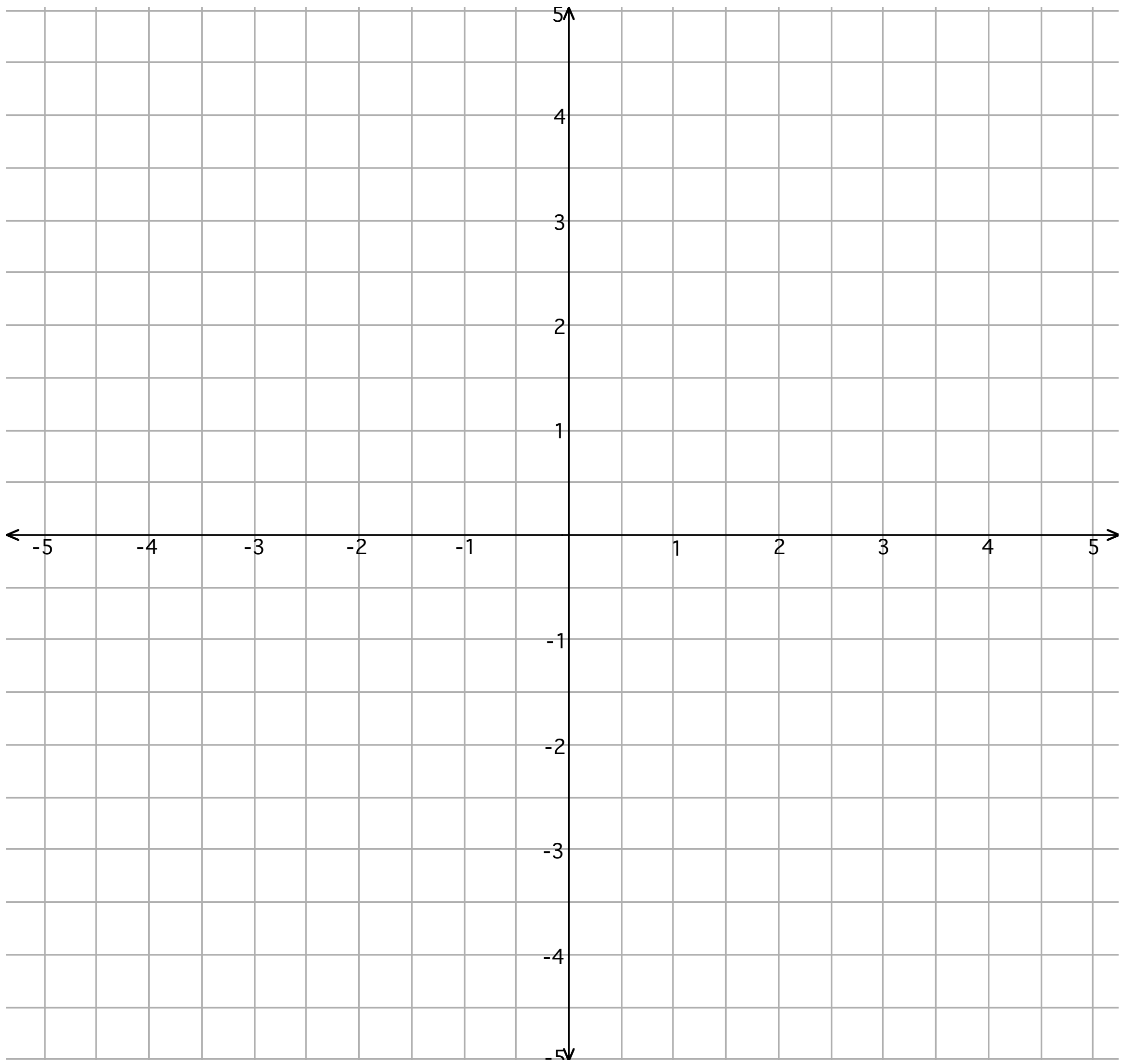


Comparison of the graphs

Both graphs are closely related to the graph of $y = x^2$ (the light dotted curves above). In the upper graph, all steps have a width of 1 unit, but they jump by increasing (upward) amounts. In the lower graph, all steps jump by 1 unit, but they are of different widths (decreasing upward). In both graphs, the dots are always attached to the graph of $y = x^2$. The lower graph is symmetric across the *y*-axis. Many additional observations are possible.

Handout 1

Coordinate grid for problems 2–6



Handout 2

Coordinate Grid for Problem #8