Deep Algebra Projects: Algebra 1 / Algebra 2
Pythagorean Triples

Topics
- The Pythagorean theorem, Pythagorean triples, and stereographic projections
- Opposites and reciprocals (additive and multiplicative inverses)
- Rational and irrational numbers
- Multiplying binomials; solving linear equations
- Equations of graphs; intersection points of graphs
- The unit circle (definition, equation, coordinates, patterns)
- Quadratic equations (solvable without factoring or using the quadratic formula)

The topics above show how the Pythagorean Triples project reinforces students' understanding of previously learned concepts and skills (opposites, reciprocals, solving linear equations, multiplying binomials) at the same that time that it sets the stage for upcoming learning (non-linear systems, quadratic equations, concepts underlying unit-circle trigonometry, etc.). And it does all of this by having students explore a well-known "recreational math" topic.

Before you begin, ensure that students understand the definition of a *Pythagorean triple*: three *natural* numbers that can be the side lengths of a right triangle. 3,4,5 is the most famous example. You can verify that it is a Pythagorean triple by checking that (1) 3, 4, and 5 are all natural numbers, and (2) they satisfy the Pythagorean theorem.
$$3^2 + 4^2 = 5^2$$

As students will learn, there is an infinite number of Pythagorean triples, and there are many ways to produce them. The method that students explore in this project makes use of *stereographic projection*—that is, using $(0,1)$ as a starting point to "project" points on a unit circle onto a real number line passing through the center of the circle. After completing their work on this activity, some students may be interested in doing further research on stereographic projections—for example, learning how they look in three dimensions and how they are used in cartography (mapmaking).

Important note: This project will work best if you wait until students have completed one problem before handing out the next one. Some problems contain "spoilers."

Stage 1

In Problem #1, students explore the expression $x^2 + y^2$ and discover that its value is always 1 on the unit circle. Use this opportunity to reinforce students' understanding of the *equation of a graph*. Emphasize that every point on the unit circle satisfies the equation $x^2 + y^2 = 1$ *and* that every point off the circle fails to satisfy it. In other words, the equation describes the *exact* set of points on the graph.

Students are likely to solve Problem #2 numerically by using their experience from Problem #1 to help them guess and test solutions. If some of them try an algebraic approach, please refer to the Problem #5 Solutions for a possible approach. Students are likely to express their solutions in decimal form, which is fine. However, at some point you may want to suggest that they try writing them as fractions as well in order to highlight connections with Pythagorean triples.

Problem #3 expands students' thinking beyond quadrant I, bringing out many beautiful patterns in the process. Consider using this opportunity to introduce them to the names of the four quadrants if they don't know them. (See the Solutions for Problem #3.)

What students should know
- Understand the meaning of the *graph of an equation*.
- Understand and apply the *slope-intercept* form of a linear equation.
- Understand and apply the Pythagorean theorem.
- Know the definition of a Pythagorean triple. Be familiar with the example 3,4,5.
- Recognize and describe reflection and rotation symmetry.
- Understand the distinction between *opposites* and *reciprocals*.

What students will learn
- Discover and understand the formula for the equation of the unit circle.
- Gain a deeper understanding of the meaning of the *equation of a graph.*
- Reinforce understanding of *slope-intercept* form of linear equations.
- Connect the Pythagorean theorem to the unit circle.
- Understand what intersection points of graphs mean in the context of equations.
- Make connections between coordinates and symmetry.
- Begin to make connections between Pythagorean triples and the unit circle.

Problem #1
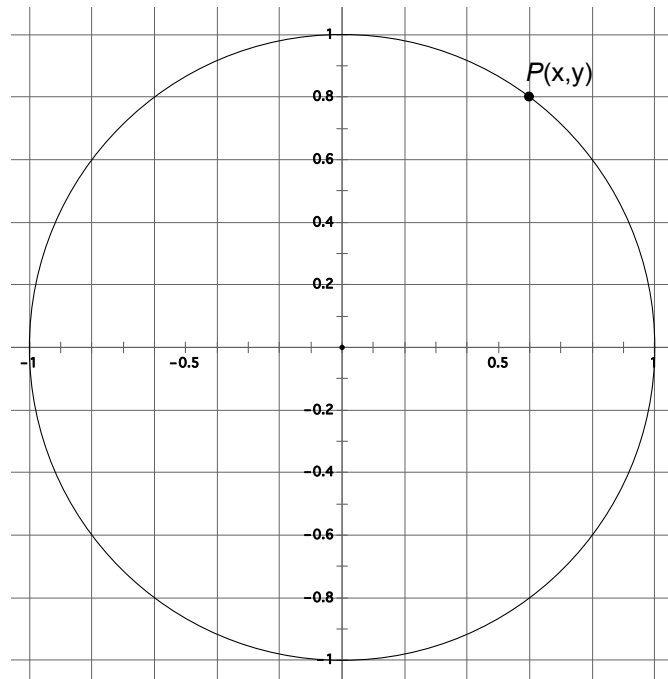
$$x^2 + y^2$$

Directions

- Estimate or calculate the value of $x^2 + y^2$.
- Slide P around the circle to some other locations. Estimate or calculate again.
- Describe any patterns you see, and explain what causes them.
- Decide if there are points *off* the circle that make your equation(s) true. Explain.

Conversation Starters for Problem #1
What do you notice? What do you wonder?

I notice that the circle has a radius of 1 unit and that its center is at the origin.

I notice that P seems to be located at the intersection of two gridlines.

I notice a few other points on the circle whose coordinates are easy to estimate.

I notice that most of the coordinates on the circle are hard to estimate precisely.

I notice that $x^2 + y^2$ is always 1 for the points whose coordinates are easy to read.

I notice that $x^2 + y^2$ is close to 1 for the points whose coordinates are harder to read.

I wonder if $x^2 + y^2$ is equal to 1 for every point on the circle.

If so, *I wonder* what causes this expression to equal 1 on the circle.

I notice that the expression $x^2 + y^2$ reminds me of the Pythagorean theorem.

I wonder if I can draw a right triangle with side lengths of x and y in the picture.

I notice that whenever x gets farther from 0, y gets closer to 0, and vice versa.

I wonder what are the coordinates of the point on the circle in the "middle" of each quadrant.

I wonder what happens to the value of $x^2 + y^2$ inside the circle.

I wonder what happens to the value of $x^2 + y^2$ outside the circle.

I wonder what the value of $x^2 + y^2$ would be on a circle of radius 2.

Solutions for #1

The value of $x^2 + y^2$ for the point shown
$$x^2 + y^2 = 0.6^2 + 0.8^2 = 1$$

The value of $x^2 + y^2$ for other points on the circle

There are many other points whose coordinates are obvious or at least appear easy to read. They occur at the top, bottom, left, and right of the circle and at points that are symmetrical (in one way or another) to the point (0.6,0.8). Calculations for these points include:

|		
$1^2 + 0^2 = 1$	$0 + 1^2 = 1$	$(-1)^2 + 0^2 = 1$
$0^2 + (-1)^2 = 1$	$0.8^2 + 0.6^2 = 1$	$(-0.6)^2 + 0.8^2 = 1$
$(-0.8)^2 + 0.6^2 = 1$	$(-0.8)^2 + (-0.6)^2 = 1$	$(-0.6)^2 + (-0.8)^2 = 1$

etc.

Continuing the process

Coordinates of other points are harder to determine precisely. Some students may become curious about the point on the circle halfway between (0.6, 0.8) and (0.8, 0.6). They may guess that it is located at (0.7, 0.7) and calculate that $0.7^2 + 0.7^2 = 0.98$. In fact, the point (0.7, 0.7) is located at the midpoint of a *line segment* joining (0.6, 0.8) and (0.8, 0.6). This midpoint is *slightly* closer to the origin than is the point on the circle.
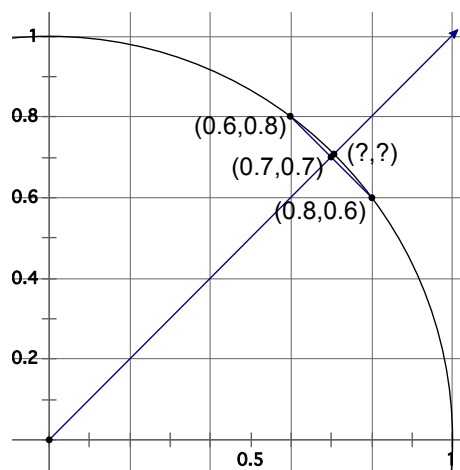
It seems clear from symmetry that the x- and y-coordiates are equal, so other reasonable estimates for the calculations may include $0.71^2 + 0.71^2 = 1.0082$ or

$0.705^2 + 0.705^2 = 0.99405$. (Students who are convinced by now that the sum should be 1 may be interested in continuing a trial and error process in order to improve their estimates. Some may develop a strategy to calculate the coordinates algebraically.)

Examples of estimates for other points include:
$$0.2^2 + 0.98^2 = 1.0004$$
$$0.9^2 + 0.42^2 = 0.9864, \text{ etc.}$$

Patterns

The evidence strongly suggests that $x^2 + y^2 = 1$ for every point on the circle. As the x-coordinate gets farther from 0, the y-coordinate appears to compensate by getting closer to 0 by just the amount needed to keep the sum constant. (Students may also comment on patterns of symmetry in the coordinates.)

Causes of the patterns

The title of the activity may provide a clue. For *any point* on the circle, you may draw a right triangle whose legs represent the x- and y-coordinates and whose hypotenuse is the radius of length 1.
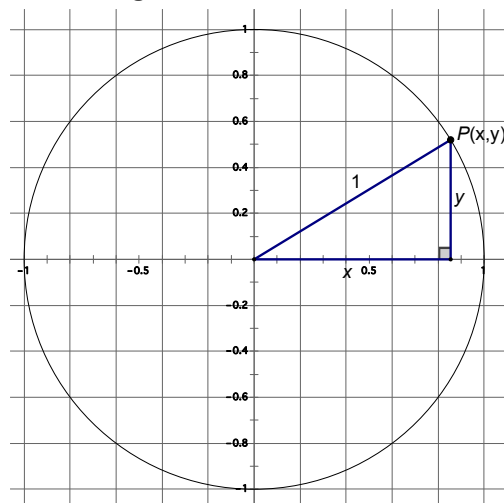

The Pythagorean theorem shows that $x^2 + y^2 = 1^2$ for every point on the circle.

Points that are not on the circle

Every point inside the circle satisfies $x^2 + y^2 < 1$, because it belongs to a right triangle whose hypotenuse is less than 1. By similar reasoning, every point outside the circle satisfies $x^2 + y^2 > 1$. In summary: every point on the circle satisfies the equation $x^2 + y^2 = 1$, and every point not on the circle fails to satisfy it. Consequently, $x^2 + y^2 = 1$ is *the equation of the circle*.

Problem #2
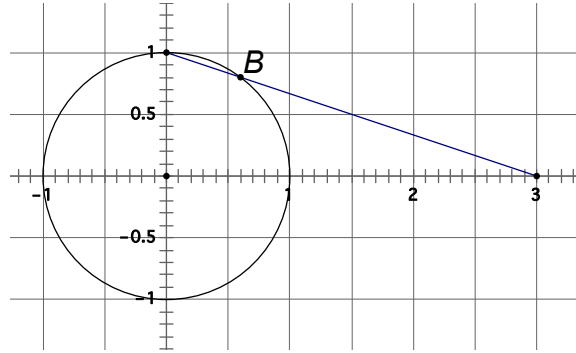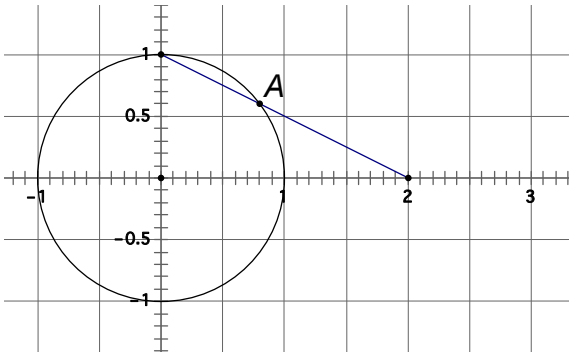
Directions

- Determine the exact coordinates of the points A and B. Justify your answers and explain your thinking processes.
- Describe a connection between your answers and Pythagorean triples.

Conversation Starters for Problem #2
What do you notice? What do you wonder?

I wonder if it would help to start by guessing coordinates.

I wonder if Problem #1 could give me any ideas.

I wonder if it would help to write the equations of the circle and the segments.

I wonder how I can tell if a point is exactly on the line segment.

I notice that it I can see the slopes and y-intercepts of the segments pretty easily.

I notice that the slope of the segment is the opposite-reciprocal of the point on the line.

I notice that both y-intercepts are the same.

I wonder if it would help to know the equation of the segment.

I wonder if it would help to write the coordinates of the points as fractions.

I notice the denominators of all of these coordinates are the same.

I wonder why this happens.

I wonder if it would help to draw right triangles inside the circle.

I notice that as the point on the number line moves away from 0, the point on the circle gets closer to (0,1)—but it will never get there.

I wonder if the intersection point will still produce a Pythagorean triple if I move the point on the number line.

I wonder if there is a way to find the points of intersection for points other than 2 or 3 on the number line (when I can't guess the coordinates).

I wonder if I could use the equations of the line and circle to find an intersection point algebraically.

Solutions for #2

The coordinates of point A

The coordinates of point A are (0.8, 0.6).

Many students may guess these coordinates based on their experience from Problem #1. In order to verify a guess, they need to check that the point belongs to both the segment and the circle. As seen in Problem #1, it belongs to the circle because the coordinates (0.8,0.6) satisfy the equation $x^2 + y^2 = 1$.

$$0.8^2 + 0.6^2 = 0.64 + 0.36 = 1$$

There are a number of ways to check that it lies on the segment—for example, by proportional reasoning (perhaps using similar triangles). Another approach is to find an equation for the segment. Since the y-intercept is 1 and the slope is $-\frac{1}{2}$, the equation is

$$y = -\frac{1}{2}x + 1.$$

The point lies on the segment, because its coordinates (0.8,0.6) satisfy this equation.

$$-\frac{1}{2}(0.8) + 1 = -0.4 + 1 = 0.6$$

Since the coordinates of A satisfy both equations, the point lies at an intersection point of their graphs.

Note: Some students may calculate the coordinates directly by solving a system of equations. This approach is discussed in Problem #5 where it is needed because it is too hard to guess coordinates.

The coordinates of point B

The coordinates of point B lie at (0.6,0.8).

Students may reason in the same way as above. The equation of the segment is

$$y = -\frac{1}{3}x + 1.$$

The coordinates (0.6,0.8) satisfy the equations for both the circle and the segment:

$$0.6^2 + 0.8^2 = 0.36 + 0.64 = 1$$

and

$$-\frac{1}{3}(0.6) + 1 = -0.2 + 1 = 0.8.$$

Again, students could also find the solution by solving a system of equations.

A connection to Pythagorean triples

The numerators and denominators of the coordinates, written as fractions, form Pythagorean triples.

For example, in simplest form, the coordinates of A and B are $\left(\frac{4}{5}, \frac{3}{5}\right)$ and $\left(\frac{3}{5}, \frac{4}{5}\right)$. In both cases, the numerators and denominators are 3, 4, and 5, which form the Pythagorean triple 3,4,5.

Important Notes: The numbers 0.6, 0.8, and 1 belong to the sides of a right triangle, but they do not form a Pythagorean triple, because they are not all positive whole numbers.

Students may not think of writing the coordinates as fractions. (You may eventually need to suggest the idea, but let them think on their own for a while first!) Even then, they may recognize a connection to the Pythagorean triple 6,8,10.

Some students may realize that either point A or B gives an *infinite* number of Pythagorean triples, because (1) the fractions may be written in other equivalent forms such as $\frac{6}{10}$ and $\frac{8}{10}$ or $\frac{9}{15}$ and $\frac{12}{15}$, etc., or (2) there is an infinite number of similar right triangles having these numerators and denominators as measurements. (These observations will arise in Problem #4, so there is no need to mention them unless students bring them up.)

A Pythagorean triple that corresponds to a pair of simplest-form fractions is called a *primitive* Pythagorean triple. For example, 3,4,5 is a primitive Pythagorean triple, but 6,8,10; 9,12,15 etc., are not.

Problem #3
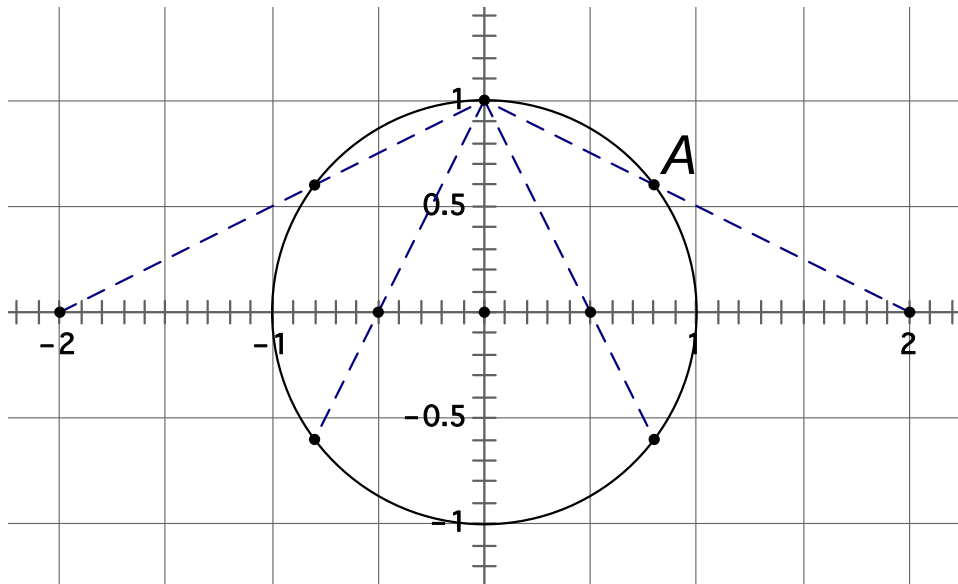
Directions

- Determine the coordinates of the marked points. Justify your answers and explain your thinking for the points on the circle.
- Find and describe as many patterns as you can.
- Create a picture like this for point B in the second picture from Problem #2. Show the coordinates of all important points, and compare your results to those for point A.

Conversation Starters for Problem #3
What do you notice? What do you wonder?

I notice that point A is the same as the point in the first picture from Problem #2.

I notice that the point inside the circle on each side of the number line is the reciprocal of the point outside the circle. (2 and $\frac{1}{2}$ are reciprocals; so are -2 and $-\frac{1}{2}$.)

I notice that the four points on the circle (not including the top one) look symmetrical.

I wonder how I can test that these points really are symmetrical.

I wonder if these patterns will continue if I start with the number 3 on the line.

I wonder if these patterns will continue if I start with a fraction on the line.

I wonder what causes these patterns.

Solutions for #3

The coordinates of the marked points

Center of the circle	$(0,0)$
Top of the circle	$(0,1)$
Outside the circle on the x-axis	$(2,0)$ and $(-2,0)$
Inside the circle on the x-axis	$(\frac{1}{2},0)$ and $(-\frac{1}{2},0)$
On the circle, upper-right quadrant (I)	$(0.8,0.6)$
On the circle, upper-left quadrant (II)	$(-0.8,0.6)$
On the circle, lower-left quadrant: (III)	$(-0.8,-0.6)$
On the circle, lower-right quadrant (IV)	$(0.8,-0.6)$

Justifications and explanations

The coordinates $(0,1)$ of the point at the top of the circle seem obvious for a circle of radius 1 centered at the origin. They were also justified in Problem #1 by showing that they satisfy the equation of the circle, $x^2 + y^2 = 1$. *Note*: This point is important, because it "projects" each point of the circle onto the number line.

The coordinates of the point, A, were discussed in Problem #1. You may obtain the coordinates of the other marked points on the circle by reflecting A across the x- and/or y-axes.

Students should probably try to justify at least one of these sets of coordinates more formally. For example, in quadrant IV, the equation of the segment is $y = -2x + 1$, and the coordinates $(0.8, -0.6)$ satisfy this equation:

$$-2(0.8) + 1 = -1.6 + 1 = -0.6.$$

Patterns

On the x-axis, the coordinates of the points inside the circle are the reciprocals of those outside the circle. Specifically, 2 and $\frac{1}{2}$ are reciprocals, and -2 and $-\frac{1}{2}$ are reciprocals.

These reciprocals project onto (or are projected from) a set of four symmetric points on the circle. The respective coordinates of the four points have the same absolute values. Every possible combination of positive / negative pairs is represented.

A picture for point B
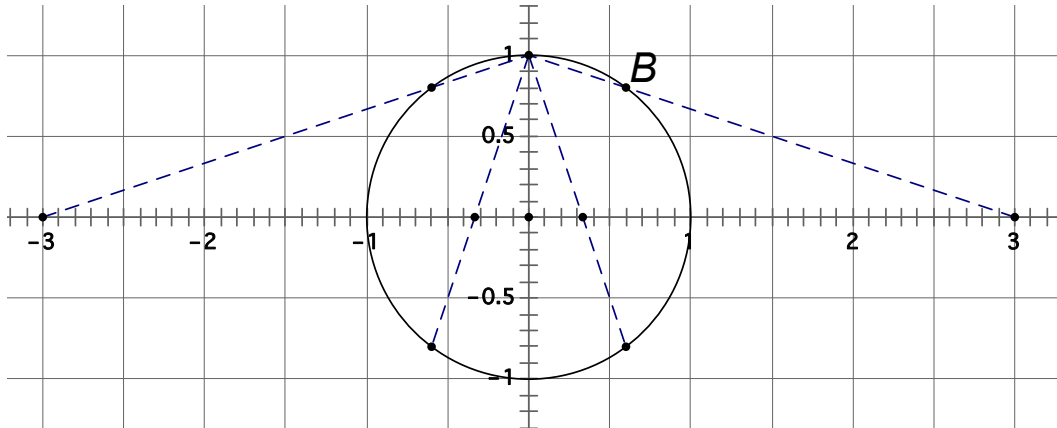
The coordinates of the marked points

Center of the circle	$(0,0)$
Top of the circle	$(0,1)$
Outside the circle on the x-axis	$(3,0)$ and $(-3,0)$
Inside the circle on the x-axis	$(\frac{1}{3},0)$ and $(-\frac{1}{3},0)$
On the circle, upper-right quadrant (I)	$(0.6,0.8)$
On the circle, upper-left quadrant (II)	$(-0.6,0.8)$
On the circle, lower-left quadrant: (III)	$(-0.6,-0.8)$
On the circle, lower-right quadrant (IV)	$(0.6,-0.8)$

Comparing to the picture for A

The general appearance of the pictures is the same. The numbers on the x-axis are still reciprocals. The numbers 0.6 and 0.8 are interchanged in the coordinates of the points on the circle.

Stage 2

In Stage 2, students who have been thinking numerically so far (which may be most of them) will begin using algebra to explore connections between Pythagorean triples and the unit circle. They begin in Problem #4 by using the equation of the unit circle to analyze a general *rational point* (a point whose coordinates are both rational numbers) on the circle. In the process, they begin to discover the importance of distinguishing between points that have rational and irrational coordinates.

Problem #5 is the heart of the activity. Before students begin, you may want to introduce the term *stereographic projection*. Imagine placing a light bulb at the point $(0,1)$ on the unit circle so that another point on the circle projects a shadow onto the number line at the point R. (See the image for Problem #5.) When you choose values of R other than 2 or 3, it is generally hard to guess the point of the circle that will project onto them. To make progress, students must solve algebraic equations related to their learning in Stage 1. I recommend that you read the Solutions to Problem #5 well in advance in order to anticipate and plan for the knowledge your students will need.

Students need not have solved systems of equations or dealt with issues such as "dividing by x," but be prepared to guide them through discussions of these ideas when the time arrives. Let them figure out as much as they can on their own first.

What students should know
- Understand concepts from Stage 1.
- Understand the definitions of rational and irrational numbers.
- Add like terms in polynomials. Multiply binomials.
- Solve linear equations.
- Be familiar with properties of similar triangles.

What students will learn
- Increase fluency solving linear equations and working with complex fractions.
- Gain a deeper understanding of rational and irrational numbers.
- Begin to explore solving quadratic equations (with constant terms of "0").
- Develop an algebraic method for finding the intersection point of two graphs.
- Understand the issues involved with "dividing by x."
- Analyze patterns in algebraic processes.

Problem #4
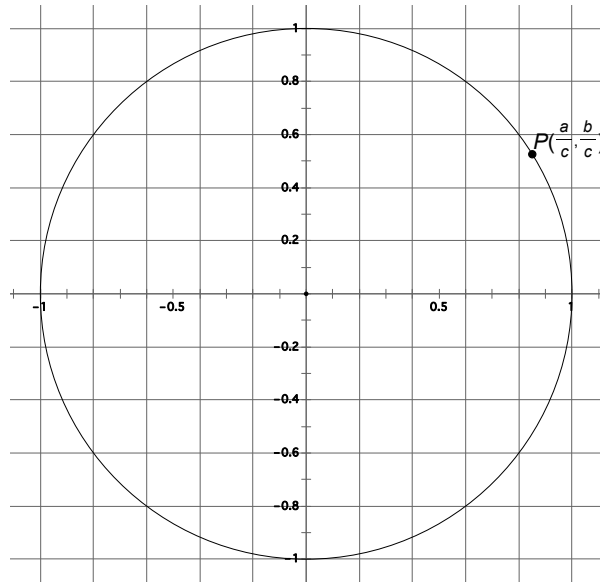
a, b, and c are integers.

Directions
- Explain or show why a, b, c must be a Pythagorean triple.
- Describe the infinite collection of Pythagorean triples related to a, b, c.
- Decide if every point on the circle leads to a Pythagorean triple. If so, explain why. If not, give an example of a point that does not.

Diving Deeper
- The diagram shows that both coordinates of P have the same denominator. Will this always happen on the unit circle when a, b, and c are integers? Explain.

Conversation Starters for Problem #4
What do you notice? What do you wonder?

I notice that I can use the point, P, to make a right triangle inside the circle.

I notice that the triangle does not have whole number side lengths.

I notice that I can make an infinite number of similar triangles from this triangle and that some of them do have whole number side lengths.

I wonder if I can use the equation of the circle that I learned about in Problem #1.

I notice that the equation of the circle looks a lot like the Pythagorean theorem.

I wonder if I would be helpful to write the equation of the circle in an equivalent form.

I wonder what would happen if I used a point on a *non*-unit circle.

I notice that some points on the circle have irrational coordinates.

I wonder if I could use these points to make Pythagorean triples.

Solutions for #4

Why a, b, c must be a Pythagorean triple

   Since the ordered pair $(\frac{a}{c}, \frac{b}{c})$ is on the *unit circle* (a circle of radius 1 centered at the origin), it satisfies the equation $x^2 + y^2 = 1$. Therefore:

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$

$$c^2\left(\frac{a^2}{c^2} + \frac{b^2}{c^2}\right) = 1 \cdot c^2$$

$$a^2 + b^2 = c^2$$

   a, b, c is a Pythagorean triple because (1) the numbers satisfy the Pythagorean theorem, and (2) all three numbers are integers.

The infinite collection of Pythagorean triples related to a, b, c

   Let n represent any positive whole number. Since a, b, and c are whole number side lengths of a right triangle, na, nb, and nc are as well. (The triangles are all similar.) Therefore, each such value of n creates a new Pythagorean triple: na, nb, nc.

   For example, 6,8,10; 9,12,15; 12,16,20... are all new Pythagorean triples related to the original one, 3,4,5.

The relationship between points on the unit circle and Pythagorean triples

   Every point of the form $(\frac{a}{c}, \frac{b}{c})$ on the unit circle leads to a Pythagorean triple for the reasons above. However, not every point on the circle may be written in this form!

   For example, if you choose $x = \frac{1}{2}$ as the x-coordinate of a point on the unit circle, you may use the Pythagorean theorem to calculate a y-coordinate of $\frac{\sqrt{3}}{2}$, which is an *irrational* number. That is, there is no way to write it as a fraction whose numerator and denominator are both integers. Therefore, it is not possible to generate the three positive *whole* numbers a, b, and c to make the Pythagorean triple.

   In summary, only the *rational points* (points whose coordinates are *both* rational numbers) on the unit circle will generate Pythagorean triples. (*Note*: The points at (1,0), (0,1), (-1,0), and (0,-1) also fail to create Pythagorean triples.)

Problem #5
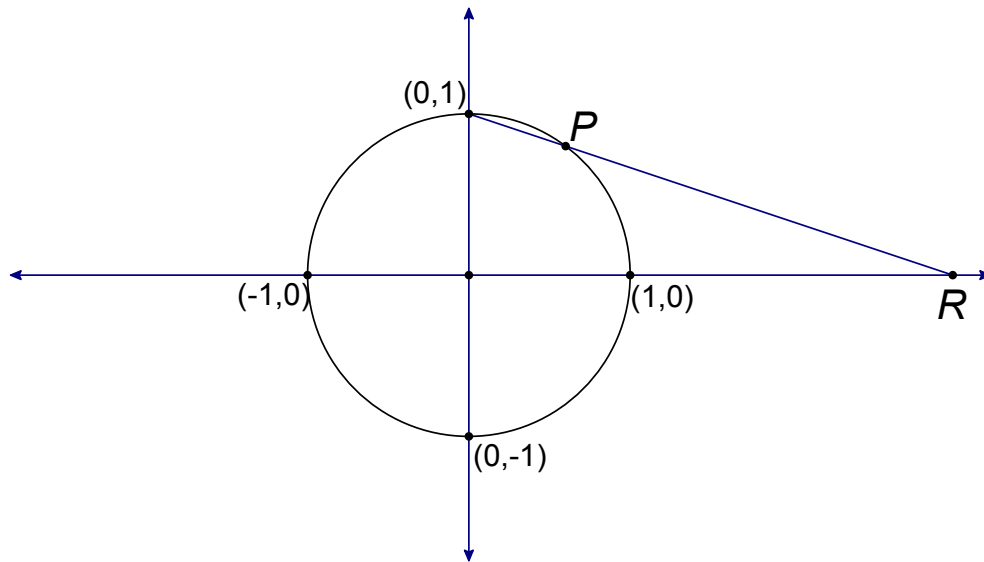
Directions

- Find the coordinates of P for R-values of 4, 1.5, and $2\frac{1}{3}$. Show your calculations, and explain your thinking.
- Describe any patterns that you notice in your calculation processes.
- Show the *primitive** Pythagorean triple associated with each value of R. Check that the numbers satisfy the Pythagorean theorem.

Diving Deeper

- Predict the total number of primitive Pythagorean triples that exist. Explain your thinking.

*The numbers in a *primitive* Pythagorean have no common factor except 1.

Conversation Starters for Problem #5
What do you notice? What do you wonder?

I notice that it is no longer easy to guess the coordinates.

I notice that I have two equations to work with, one for the circle and one for the segment.

I notice that the coordinates of the intersection must satisfy both of these equations.

I wonder if it would still be useful to find and use an equation for the segment.

I wonder if I could guess and test to find numbers that satisfy both equations at the same time.

I notice that it is hard to solve the equations because they have two variables.

I wonder if I could combine the two equations into one equation with one variable.

I notice that my (single) equation cannot be solved using only the methods I have learned for linear equations.

I notice there will be a "middle" (linear) term when I square the binomial.

I notice that I can make the equation look simpler by combining like terms.

I notice that the constant terms are always just 1, and they always subtract away!

I notice that it would help to divide both sides by x. *I wonder* if it is okay to do this.

I wonder how I would solve this kind of equation if the constant terms did not subtract away.

I wonder how I can find the y value after I find the x value.

I notice that I should get the same y value for both equations.

Solutions for #5

The coordinates of P

For $R = 4$ $\qquad$ $(\frac{8}{17}, \frac{15}{17})$

For $R = 1.5$ $\qquad$ $(\frac{12}{13}, \frac{5}{13})$

For $R = 2\frac{1}{3}$ $\qquad$ $(\frac{21}{29}, \frac{20}{29})$

Sample calculations and explanations for $R = 4$

The equation of the line segment through (4,0) and (0,1) is $y = -\frac{1}{4}x + 1$.

The equation of the unit circle is $x^2 + y^2 = 1$.

The intersection point, (x, y), in quadrant I must satisfy both equations.

One way to obtain a single equation in one variable is to substitute the linear expression for y into the second equation.

$$x^2 + \left(-\frac{1}{4}x + 1\right)^2 = 1$$

Solve for x in order to find the x-coordinate of the point.

$$x^2 + \frac{1}{16}x^2 - \frac{1}{4}x - \frac{1}{4}x + 1 = 1$$

$$\frac{17}{16}x^2 - \frac{1}{2}x = 0$$

$$\frac{17}{16}x^2 = \frac{1}{2}x$$

$x = 0$ satisfies this equation and leads to the solution $x = 0, y = 1$ which appears at the top of the circle as the point (0,1). If $x \neq 0$, then you may divide both sides by x to find the x-coordinate of the other solution.

$$\frac{17}{16}x = \frac{1}{2}$$

$$x = \frac{1}{2} \cdot \frac{16}{17} = \frac{8}{17}$$

To find the corresponding value of y, substitute this value of x into either of the original equations. Using the linear equation:

$$y = -\frac{1}{4} \cdot \frac{8}{17} + 1 = -\frac{2}{17} + 1 = \frac{15}{17}$$

The coordinates of the point are $(\frac{8}{17}, \frac{15}{17})$. You may check that this pair satisfies the equation $x^2 + y^2 = 1$ as well.

Some students may find it helpful to draw a graph in order to check that these coordinates seem reasonable.

Sample calculations and explanations for $R = 1.5$

The equation of the line segment through $(1.5, 0)$ and $(0,1)$ is $y = -\frac{2}{3}x + 1$.

The equation of the unit circle is $x^2 + y^2 = 1$.

$$x^2 + \left(-\frac{2}{3}x + 1\right)^2 = 1$$

$$x^2 + \frac{4}{9}x^2 - \frac{2}{3}x - \frac{2}{3}x + 1 = 1$$

$$\frac{13}{9}x^2 - \frac{4}{3}x = 0$$

$$\frac{13}{9}x^2 = \frac{4}{3}x$$

When $x \neq 0$:

$$\frac{13}{9}x = \frac{4}{3}$$

$$x = \frac{4}{3} \cdot \frac{9}{13} = \frac{12}{13}$$

Substituting this value for x in one of the original equations:

$$y = -\frac{2}{3} \cdot \frac{12}{13} + 1 = -\frac{8}{13} + 1 = \frac{5}{13}$$

Therefore, the coordinates of the point are $(\frac{12}{13}, \frac{5}{13})$.

Sample calculations and explanations for $R = 2\frac{1}{3}$

The equation of the line segment through $(2\frac{1}{3}, 0)$ and $(0,1)$ is $y = -\frac{3}{7}x + 1$.

The equation of the unit circle is $x^2 + y^2 = 1$.

$$x^2 + \left(-\frac{3}{7}x + 1\right)^2 = 1$$

$$x^2 + \frac{9}{49}x^2 - \frac{3}{7}x - \frac{3}{7}x + 1 = 1$$

$$\frac{58}{49}x^2 - \frac{6}{7}x = 0$$

$$\frac{58}{49}x^2 = \frac{6}{7}x$$

When $x \neq 0$:

$$\frac{58}{49}x = \frac{6}{7}$$

$$x = \frac{6}{7} \cdot \frac{49}{58} = \frac{21}{29}$$

Substituting this value for x in one of the original equations:

$$y = -\frac{3}{7} \cdot \frac{21}{29} + 1 = -\frac{9}{29} + 1 = \frac{20}{29}$$

Therefore, the coordinates of the point are $\left(\frac{21}{29}, \frac{20}{29}\right)$.

Examples of patterns in the calculations
- The slope of the line segment is always the opposite reciprocal of R.
- The 1s always subtract away.
- $(0,1)$ is always a solution to both equations.
- In the second from the last step for calculating x, the constant fraction term (shown on the right side of the equation) is always 2 times the reciprocal of R.
- In the last step (where you multiply two fractions to calculate x), one numerator is always the square of the other denominator.

Many other observations are possible.

The primitive Pythagorean triples

$R = 4$

    Pythagorean triple: 8,15,17

    Check: $8^2 + 15^2 = 64 + 225 = 289$ $17^2 = 289$ ✔

$R = 1.5$

    Pythagorean triple: 5,12,13

    Check: $5^2 + 12^2 = 25 + 144 = 169$ $13^2 = 169$ ✔

$R = 2\frac{1}{3}$

    Pythagorean triple: 20,21,29

    Check: $20^2 + 21^2 = 400 + 441 = 841$ $29^2 = 841$ ✔

Problem #6

A New Pythagorean Triple: 12, 35, 37

Directions
- Name the coordinates of two points in Quadrant I that lead to this Pythagorean triple. Find the value of R for each one. Show and explain your thinking.
- Find 6 more points on the unit circle that relate to the same Pythagorean triple, and find their values of R. Show and explain your thinking.

Conversation Starters for Problem #6
What do you notice? What do you wonder?

I notice that this problem involves thinking backwards from the way I did in Problem #5.

I wonder if finding an equation for the segment could help me find values for R.

I notice that I can use $(0,1)$ along with the coordinates of the other point on the circle to find an equation for the segment.

I notice that the value of R shows up as $(R, 0)$ on the number line. That is, $x = R$ and $y = 0$.

I notice that I could have predicted the values of R by remembering a pattern that I discovered earlier: the slope of the segment is the opposite reciprocal of the value of R.

I notice a connection to Problem #3 (in looking for the other six points).

I notice that I can use patterns to find the other six points and their R values.

I wonder how I can check that the patterns give the correct results.

> You could do the calculations directly: choose symmetric points on the circle in other quadrants and calculate their R values to check that they are reciprocals and/or opposites of the first quadrant R values. Or you could do the reverse: choose opposite and/or reciprocal R values, calculate the points that project onto them, and ensure that the points are symmetrical to the original first-quadrant points.

Solutions for #6

Two points in Quadrant I that lead to the Pythagorean triple 12,35,37

    The two points in Quadrant I leading to the Pythagorean triple 12,35,37 are $(\frac{12}{37}, \frac{35}{37})$ and $(\frac{35}{37}, \frac{12}{37})$.

The R values associated with these points

    The R values associated with these points are 6 and $\frac{7}{5}$, respectively.

For $(\frac{12}{37}, \frac{35}{37})$

    The line segment through the points $(0,1)$ and $(\frac{12}{37}, \frac{35}{37})$ has a slope of

$$\frac{1 - \dfrac{35}{37}}{0 - \dfrac{12}{37}} = \frac{\dfrac{2}{37}}{-\dfrac{12}{37}} = -\frac{2}{12} = -\frac{1}{6}.$$

    Therefore, its equation is

$$y = -\frac{1}{6}x + 1.$$

    The value of R is the x-coordinate of the point on this segment where $y = 0$.

$$0 = -\frac{1}{6}R + 1$$

$$\frac{1}{6}R = 1$$

$$R = 6$$

For $(\frac{35}{37}, \frac{12}{37})$

    The line segment through the points $(0,1)$ and $(\frac{35}{37}, \frac{12}{37})$ has a slope of

$$\frac{1 - \dfrac{12}{37}}{0 - \dfrac{35}{37}} = \frac{\dfrac{25}{37}}{-\dfrac{35}{37}} = -\frac{25}{35} = -\frac{5}{7}.$$

    Therefore, its equation is

$$y = -\frac{5}{7}x + 1.$$

    The value of R is the x-coordinate of the point on this segment where $y = 0$.

$$0 = -\frac{5}{7}R + 1$$

$$\frac{5}{7}R = 1$$

$$R = \frac{7}{5}$$

6 more points and values of R for $12, 35, 37$

The original points:

Quadrant I

$\left(\frac{12}{37}, \frac{35}{37}\right)$ $R = 6$

$\left(\frac{35}{37}, \frac{12}{37}\right)$ $R = \frac{7}{5}$

The six new points:

Quadrant II

$\left(-\frac{12}{37}, \frac{35}{37}\right)$ $R = -6$

$\left(-\frac{35}{37}, \frac{12}{37}\right)$ $R = -\frac{7}{5}$

Quadrant III

$\left(-\frac{12}{37}, -\frac{35}{37}\right)$ $R = -\frac{1}{6}$

$\left(-\frac{35}{37}, -\frac{12}{37}\right)$ $R = -\frac{5}{7}$

Quadrant IV

$\left(\frac{12}{37}, -\frac{35}{37}\right)$ $R = \frac{1}{6}$

$\left(\frac{35}{37}, -\frac{12}{37}\right)$ $R = \frac{5}{7}$

In quadrant II, you reflect the points in quadrant I across the y-axis by taking the opposite values of R. (This makes the x-coordinates negative.)

In quadrant IV, you reflect the points in quadrant I across the x-axis by taking the reciprocal values of R. (This makes the y-coordinates negative.)

In quadrant III, you reflect the points in quadrant I across each axis in succession (in either order) by taking the opposite-reciprocal values of R. Equivalently, you rotate the points 180° about the origin. (This makes all of the coordinates negative.)

Stage 3

In Stage 3, students generalize their discoveries from earlier in the exploration. Specifically, they discover algebraic formulas to (1) find the values of R associated with a certain Pythagorean triple, and (2) use arbitrary rational numbers to generate Pythagorean triples. The latter discovery takes the form of a famous set of three expressions, one each for a, b, and c.

The processes that students use will be essentially the same as those in Problems #5 and #6, but the calculations are more challenging, because everything is written in terms of variables. Once students find the formulas, they will test them against numeric results that they obtained earlier in the exploration.

What students should know
- Understand concepts and skills from Stages 1 and 2.

What students will learn
- Use variables to generalize numeric results.
- Discover and apply methods for adding algebraic fractions (Problem #8).
- Discover, use, and justify a set of well-known formulas for generating Pythagorean triples.
- Recognize and justify patterns and relationships related to Pythagorean triples.

Problem #7

a, b and c are integers.
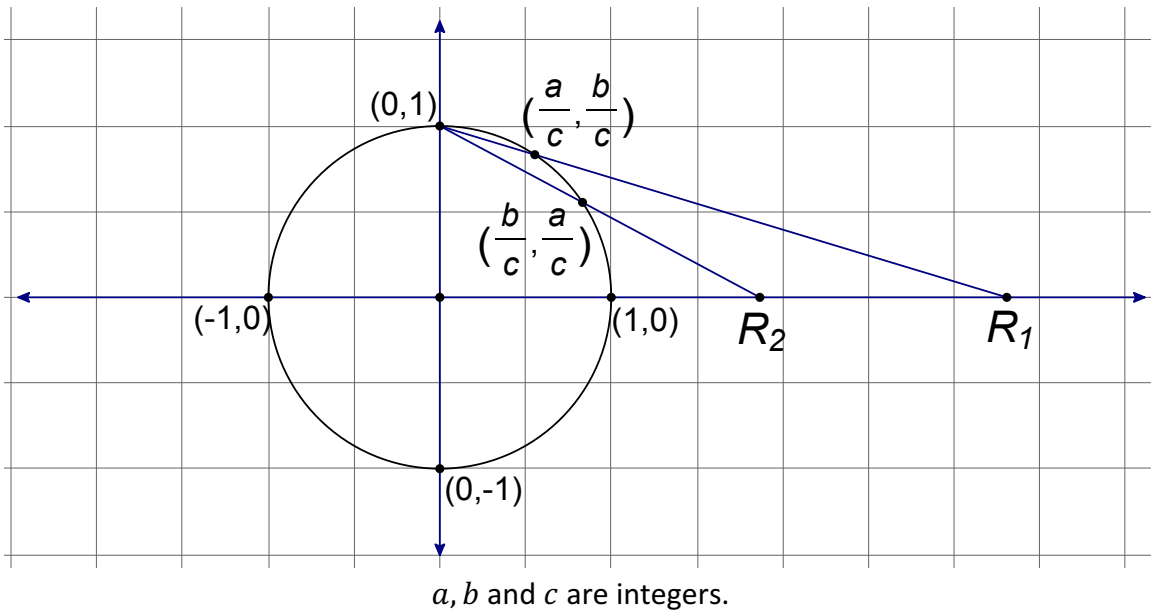
Directions

- Find expressions for the two values, R_1 and R_2, associated with the Pythagorean triple a, b, c in the picture above. Explain your thinking.
- Test your expressions using the values from Problem #6.
- Find a quick way to predict R_2 from R_1 (and vice versa). Explain your thinking.

Diving Deeper

- Prove that your method for predicting predict R_2 from R_1 is correct. (This may be easier to do after you complete Problem #8.)
- Prove that you can use exactly the same method to predict R_2 from R_1!

Conversation Starters for Problem #7
What do you notice? What do you wonder?

I wonder if I can use the same kind of process as in Problem #6.

I notice that I can use $(0,1)$ and the algebraic coordinates of the other points on the circle to find equations for the segments.

*I notice* that the coefficient of x is now an algebraic expression.

I notice that I could have predicted the R values by remembering a pattern that I discovered earlier: the slope of the segment is the opposite reciprocal of the value of R.

I notice that the two expressions for R look almost the same. (The only difference is that a and b are interchanged.)

I notice that multiplying a, b, and c by the same counting number does not change the value of R.

I notice that all R values for points in the first quadrant are greater than 1.

I notice an R-value "balance point" (B) that appears to be slightly greater than 2.4.
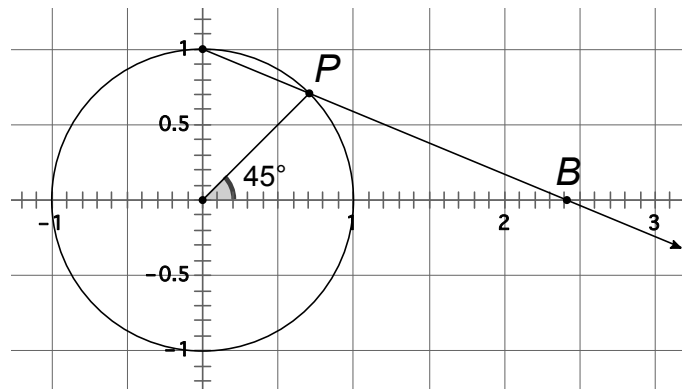
I notice that every pair of R values (when P is in Quadrant I) for the same Pythagorean triple contains one value between 1 and B and one value greater than B.

I wonder what the exact value of B is.

    The exact value of B is $1 + \sqrt{2} \approx 2.41$. In order to calculate it, students could find the coordinates of P (which are $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$) and project it onto the number line. (Unfortunately, this point does not produce a Pythagorean triple, because its coordinates are irrational.)

Solutions for #7

Expressions for R_1 and R_2

$$R_1 : \frac{a}{c-b} \qquad\qquad R_2 : \frac{b}{c-a}$$

A thinking process for R_1

The slope of the line containing $(0,1)$ and $(\frac{a}{c}, \frac{b}{c})$ is

$$\frac{1-\frac{b}{c}}{0-\frac{a}{c}} \cdot \frac{c}{c} = \frac{c-b}{0-a} = \frac{c-b}{-a} \cdot \frac{-1}{-1} = \frac{b-c}{a}$$

Therefore, the equation of the segment is

$$y = \frac{b-c}{a}x + 1$$

R_1 is the x-coordinate when $y=0$. Therefore,

$$0 = \frac{b-c}{a}R_1 + 1$$

$$\frac{b-c}{a}R_1 = -1$$

$$R_1 = -1 \cdot \frac{a}{b-c}$$

$$R_1 = \frac{a}{c-b}$$

A thinking process for R_2

Since a and b are swapped in the original fractions, they will also be swapped in the equation. Therefore,

$$R_2 = \frac{b}{c-a}.$$

Alternatively, you could repeat the process used for R_1.

Note: The second equation is not really new. Both equations express the fact that numerator of R is always the numerator of the x-coordinate, and the number subtracted from c is always the numerator of the y-coordinate. Writing the formulas separately just shows the *relationship* between the two R values.

Testing the expressions using the values from Problem #6

The values from Problem #6 are shown here for reference:

1. $\left(\frac{12}{37}, \frac{35}{37}\right)$ $R_1 = 6$
2. $\left(\frac{35}{37}, \frac{12}{37}\right)$ $R_2 = \frac{7}{5}$

For these examples, $a = 12, b = 35,$ and $c = 37.$

Example 1

$$R_1 = \frac{a}{c-b} = \frac{12}{37-35} = \frac{12}{2} = 6 \quad \checkmark$$

Example 2

$$R_2 = \frac{b}{c-a} = \frac{35}{37-12} = \frac{35}{25} = \frac{7}{5} \quad \checkmark$$

A quick way to predict R_2 from R_1

If $R_1 = k$, then $R_2 = \frac{k+1}{k-1}$.

Some students might guess this relationship from the single pair of R values above. However, one example is not nearly enough to be confident. You may want to suggest that students search for other pairs using specific triples from earlier in the activity:

$$3,4,5 \quad 8,15,17 \quad 5,12,13 \quad 20,21,29$$

Using the formulas that they just created for R_1 and R_2, they may calculate:

3,4,5	$R_1 = 3$ $R_2 = 2$	8,15,17 $R_1 = 4$ $R_2 = \frac{5}{3}$
5,12,13	$R_1 = 5$ $R_2 = \frac{3}{2}$	20,21,29 $R_1 = \frac{5}{2}$ $R_2 = \frac{7}{3}$

The prediction obviously works for 8,15,17. You can quickly see that it works for 3,4,5 and 5,12,13 by simplifying the fraction that you get after using $\frac{k+1}{k-1}$. Showing that it works for 20,21,29 takes more effort:

$$\frac{\frac{5}{2}+1}{\frac{5}{2}-1} = \frac{\frac{7}{2}}{\frac{3}{2}} = \frac{7}{3}$$

It doesn't matter whether you start with R_1 or R_2! Substituting *either* value into $\frac{k+1}{k-1}$ will produce the other value. Can you check this? Can you prove it?

Problem #8
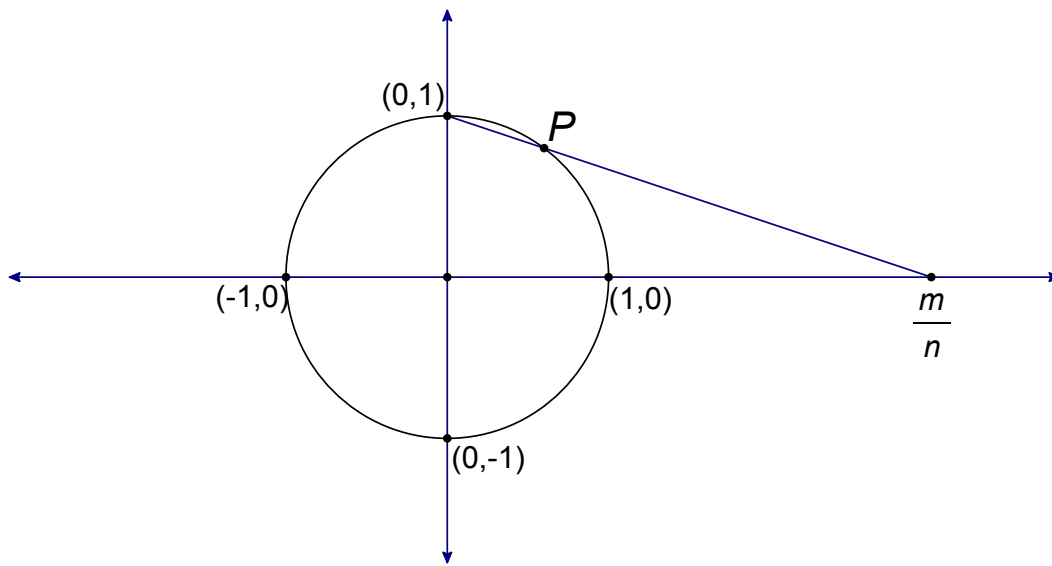
m and n are non-zero integers.

Directions

- Find an expression for each number in the Pythagorean triple related to $R = \frac{m}{n}$.
- Test your expressions using values of R from earlier problems in the activity.
- Prove that your expressions satisfy the Pythagorean theorem.

Conversation Starters for Problem #8
What do you notice? What do you wonder?

I notice that I need to find expressions for the coordinates of P first.

I wonder if I can use the same process that I used in Problem #5.

I notice that the x-coefficient in my linear equation is an algebraic expression instead of a number.

I notice that the calculation patterns that I discovered in Problem #5 show up again in the variables!

    Specifically:

- The slope of the line segment and the value of R are still opposite reciprocals.
- The 1s still subtract away.
- The constant fraction term (shown in the Solutions on the right side of the equation in the third-to-last step) is still 2 times the reciprocal of R.
- In the second-to-last step (where you multiply two fractions to calculate x), one numerator is still the square of the other denominator.

I wonder if my formulas will verify that the two R values from the same Pythagorean triple will both produce that triple.

Solutions for #8

An expression for each number in the Pythagorean triple related to $R = \dfrac{m}{n}$

$a = 2mn$

$b = m^2 - n^2$

$c = m^2 + n^2$

Note: These formulas for producing a Pythagorean triple from a rational number are famous, but many people do not know where they come from. That's a big part of what you discover in this problem!

A strategy for calculating the expressions for $a, b,$ and c

Find coordinates for P (written as fractions). The numerator of the x-coordinate will be a; the numerator of the y-coordinate will be b; and the denominators (which should be equal) will be c. The process will be almost the same as with numbers.

The slope of the line segment joining $(0,1)$ and $\left(\dfrac{m}{n}, 0\right)$ is

$$\frac{1-0}{0-\dfrac{m}{n}} = \frac{1}{-\dfrac{m}{n}} = -\frac{n}{m}.$$

The equation of the segment is

$$y = -\frac{n}{m}x + 1.$$

Substituting this expression for y into the unit-circle equation, $x^2 + y^2 = 1$, gives

$$x^2 + \left(-\frac{n}{m}x + 1\right)^2 = 1$$

$$x^2 + \frac{n^2}{m^2}x^2 - \frac{n}{m}x - \frac{n}{m}x + 1 = 1$$

$$\left(1 + \frac{n^2}{m^2}\right)x^2 - \frac{2n}{m}x = 0$$

$$\left(\frac{m^2 + n^2}{m^2}\right)x^2 = \frac{2n}{m}x$$

When $x \neq 0$:

$$\left(\frac{m^2 + n^2}{m^2}\right)x = \frac{2n}{m}$$

$$x = \frac{2n}{m} \cdot \frac{m^2}{m^2 + n^2}$$

$$x = \frac{2mn}{m^2 + n^2}$$

Since this is the expression for the x-coordinate, the numerator must be a and the denominator must be c. Therefore:

$$a = 2mn \text{ and } c = m^2 + n^2.$$

To find b, you need to know the expression for the y-coordinate. You may substitute the expression for x into either of the original equations. Using the linear equation:

$$y = -\frac{n}{m}x + 1$$

$$y = -\frac{n}{m}\left(\frac{2mn}{m^2 + n^2}\right) + 1$$

$$y = \frac{-2n^2}{m^2 + n^2} + \frac{m^2 + n^2}{m^2 + n^2}$$

$$y = \frac{-2n^2 + m^2 + n^2}{m^2 + n^2}$$

$$y = \frac{m^2 - n^2}{m^2 + n^2}$$

The numerator of this expression is the value of b.

$$b = m^2 - n^2$$

The denominator is the same as it was before (which it should be). Putting everything together:

$$a = 2mn \qquad b = m^2 - n^2 \qquad c = m^2 + n^2$$

Testing the expressions with values from earlier

Values from Problem #5

$R = 4$ gave a Pythagorean triple of 8,15,17

$R = 1.5$ gave a Pythagorean triple of 5,12,13

$R = 2\frac{1}{3}$ gave a Pythagorean triple of 20,21,29

Let's see if the formulas give the correct triples!

For $R = 4$ $m = 4$ $n = 1$
    $a = 2mn = 2 \cdot 4 \cdot 1 = 8$
    $b = m^2 - n^2 = 4^2 - 1^2 = 16 - 1 = 15$
    $c = m^2 + n^2 = 4^2 + 1^2 = 16 + 1 = 17$ ✔

For $R = 1.5$ (or $\frac{3}{2}$) $m = 3$ $n = 2$
    $a = 2mn = 2 \cdot 3 \cdot 2 = 12$
    $b = m^2 - n^2 = 3^2 - 2^2 = 9 - 4 = 5$
    $c = m^2 + n^2 = 3^2 + 2^2 = 9 + 4 = 13$ ✔

For $R = 2\frac{1}{3}$ (or $\frac{7}{3}$) $m = 7$ $n = 3$
    $a = 2mn = 2 \cdot 7 \cdot 3 = 42$
    $b = m^2 - n^2 = 7^2 - 3^2 = 49 - 9 = 40$
    $c = m^2 + n^2 = 7^2 + 9^2 = 49 + 9 = 58$ ✔

Notice that:
- a and b may be reversed. (Remember that they are interchangeable. For example, 12,5,13 is the same Pythagorean triple as 5,12,13.)
- You may not get the *primitive* form of the Pythagorean triple. You can "simplify" it if you like. (For example, 42,40,58 simplifies to 21,20,29, which agrees with the example from Problem #5.)

What happens if you choose the other value of R for a Pythagorean triple? For example, $R = 4$ and $R = \frac{5}{3}$ should *both* produce the same triple: 8,15,17.

For $R = \frac{5}{3}$ $m = 5$ $n = 3$
    $a = 2mn = 2 \cdot 5 \cdot 3 = 30$
    $b = m^2 - n^2 = 5^2 - 3^2 = 25 - 9 = 16$
    $c = m^2 + n^2 = 5^2 + 3^2 = 25 + 9 = 34$ ✔

At first, the result looks different, but 30,16,34 simplifies to 15,8,17 which is the same triple as 8,15,17—as expected!

Proving that the expressions satisfy the Pythagorean theorem

$a^2 + b^2 =$

$(2mn)^2 + (m^2 - n^2)^2 =$

$4m^2n^2 + m^4 - m^2n^2 - m^2n^2 + m^4 =$

$m^4 + 2m^2n^2 + m^4$

$c^2 =$

$(m^2 + n^2)^2$

$m^4 + m^2n^2 + m^2n^2 + m^4 =$

$m^4 + 2m^2n^2 + m^4$

Since $a^2 + b^2$ and c^2 both equal the same expression, they are equal to each other!