Deep Algebra Projects: Algebra 1 / Algebra 2
More or Less the Same

Topics

- Interpreting slopes and y-intercepts of linear functions
- Solving linear equations
- Solving linear inequalities
- Solving equations to find intersection points of graphs
- Representing solutions to algebraic inequalities on a number line

"More or Less the Same" (> , < , =) is designed to get students thinking deeply about algebraic inequalities and equations by having them examine relationships between graphs, equations, numbers, and real-world situations. Rather than giving students a specific real-world scenario to work with, this project asks students to develop their own stories that fit some mathematical information and then to use their stories to analyze the mathematics involved.

Students are assumed to understand slopes and y-intercepts of linear graphs. They will be asked to calculate them and interpret their meaning in context. The activity is also written for students who have some previous experience solving algebraic inequalities. However, you could begin the activity before teaching these procedures. Students could develop and compare their own solution methods first. When you then introduce the procedures formally, students will be able to rely on this experience to make better sense of the steps that you teach.

Similarly, students who have already learned to solve equations in order to find the point where graphs of lines intersect will have a chance to apply these skills (in Problem #2). Again, those who have not previously learned the procedures will be challenged to come up with their own strategies. Given the level of thinking required, this may a great time to let them collaborate!

More or Less the Same
Grade 8 Extension Project

Stage 1

Problems #1 and #2 "set the stage" for upcoming work on inequalities in Stage 2. Students explore a pair of graphs by creating a real-world (or imaginative) scenario for them, analyzing the meaning of slopes and y-intercepts, and writing and solving equations.

Notice that the variable t (which suggests "time") appears on the vertical axis while D, which often represents distance, appears on the horizontal one. This reversal of traditional choices requires students to think a little more carefully about the possible interpretations of the graphs and their features (especially the slopes). Of course, students may make their own choices about what the letters stand for, but using time on the vertical axis should lead to some interesting and enlightening thinking and discussion!

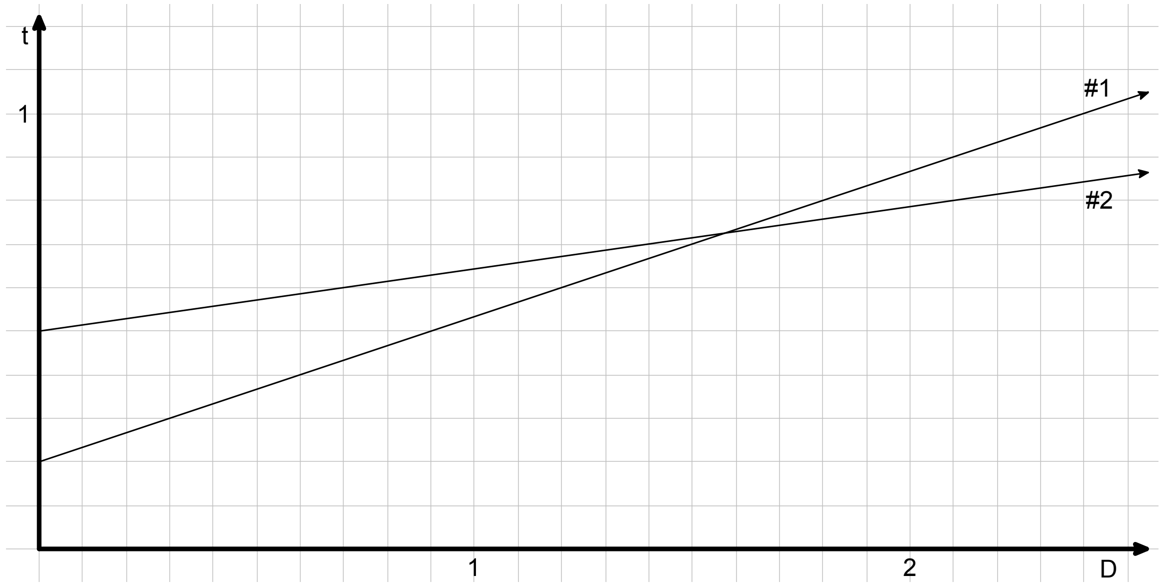
What students should know:

- Find values of slopes and intercepts from graphs.
- Understand the meanings of slopes and intercepts in linear relationships.
- Write equations of lines given their graphs (possibly using $y = mx + b$ form).
- Solve linear equations.

What students will learn:

- Analyze the connection between linear equations and intersections of graphs.
- Understand the relationship between the units of a slope and the units of its reciprocal.
- Deepen understanding of linear relationships by exploring them from many perspectives: graphs, equations, and real-world contexts.

Problem #1



Directions

- Create a real-world (or imaginative) story for the graphs.
- Find the slopes of the lines, and explain what they mean in your story.
- Explain what the reciprocals of the slopes mean in your story.
- Find the t-intercepts of the graphs, and explain what they mean in your story.

Conversation Starters for #1

What do you notice? What do you wonder?

I notice that graph #1 starts lower but climbs more quickly than graph #2.

I notice that t is on the vertical axis, which is unusual, because t usually stands for time, and time is usually on the horizontal axis.

I wonder if it still makes sense to let t stand for time?

I wonder if (or how) it will change my story to have time on the vertical axis.

I notice that the directions use the phrase t-intercept instead of y-intercept.

I wonder what D should stand for?

Distance would be a traditional choice, but students may make other choices.

I notice that the scale is small: the sides of each square on the grid are 0.1 units long.

I wonder what real-world quantities would have such small values.

I wonder if I can write the slope without using decimals in the numerator and denominator.

I notice that both of the slopes are less than 1.

I wonder why one of the questions asks about reciprocals.

I notice that the reciprocals of the slopes are whole numbers.

Solutions for #1

A sample story for the graphs

Karen stretches for 0.2 hours (12 minutes) before going on a brisk walk. Itsuko stretches for 0.5 hours (30 minutes) before beginning her jog. The input, D, represents the distance in miles that each girl travels, and t stands for the total amount of time in hours that each girl spends on her workout.

Note: Some students may create (or you may suggest that they create) stories in which more time spent preparing for some task makes the work more efficient, resulting in less time needed to complete the task. In this case, D may stand for a quantity other than distance—for example, the amount of some product that is produced.

The slopes of the lines

Line #1 has a slope of $\frac{1}{3}$, meaning that Karen takes $\frac{1}{3}$ of an hour (20 minutes) to walk 1 mile. Line #2 has a slope of $\frac{1}{7}$, meaning that Itsuko takes $\frac{1}{7}$ of an hour (about 8 and ½ minutes) to jog 1 mile. The units of the slopes are hours per mile.

Notes: Students may reason that $\frac{0.1}{0.3} = \frac{1}{3}$, etc. Also, the units of the slopes feel may feel a little strange, because time is the output. The graphs suggest the question "How many hours does it take to travel D miles?" which leads to units of hours per mile rather than miles per hour.

The reciprocals of the slopes

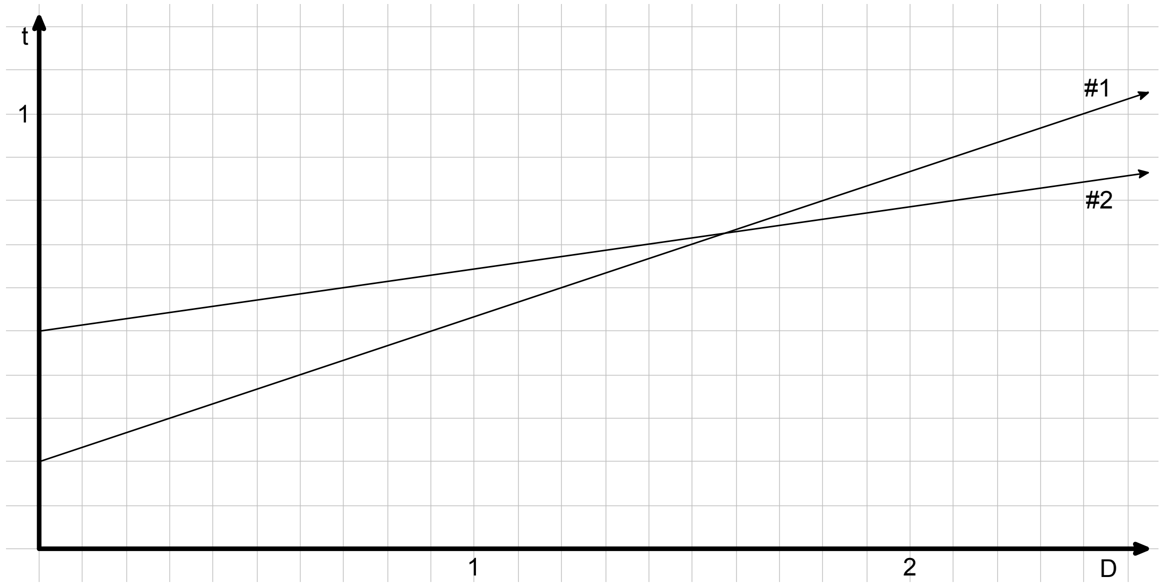
Line #1 has a slope whose reciprocal is 3, meaning that Karen travels 3 miles in 1 hour (3 miles per hour). Line #2 has a slope whose reciprocal is 7, meaning that Itsuko travels 7 miles in 1 hour (7 miles per hour).

If time were the input, the slopes would *be* these reciprocals (3 and 7), and the units would be miles per hour, which may feel more natural. Why might it sometimes make sense to choose time as the output anyway?

The t-intercepts

The t-intercepts of the graphs are 0.2 for graph #1 and 0.5 for graph #2. They stand for the amount of time that each girl stretches before beginning to walk or jog.

Problem #2



Directions

- Estimate the point where the graphs intersect.
- Create an equation for each graph*. Explain your thinking.
- Use your equations to calculate the point where the graphs intersect.
- Explain what the intersection point means in your story.

*Suggestion: Use t_1 for the t value of graph #1 and t_2 for the t value of graph #2.

Conversation Starters for #2

What do you notice? What do you wonder?

I notice that the graphs intersect at a point that is hard to read precisely.

I notice that the scales allow me to estimate values to about the nearest hundredth.

I wonder if making a table would help me find the equations.

I wonder if there is a formula or some other shortcut for finding the equations.

I wonder if it will work better to use fractions or decimals in my equations.

I notice that I can test my equations by substituting values for t and D.

I wonder why solving my equation tells me the point of intersection.

I notice that having fractions and decimals in the equations makes them a little harder to solve.

I notice that having fractions and decimals in the equations does not force me to change my *method* for solving them.

I wonder if I could find equivalent equations that don't contain fractions?

I notice that I can test my solution for t by substituting my value for D into both formulas and checking that I get the same answer both times.

I notice that my solutions for t and D agree (or not) with my estimates for the coordinates of the point of intersection.

Solutions for #2

Estimating the point of intersection

The point of intersection appears to lie at about (1.57, 0.73). Given the scales on the axes, these numbers should be within about 1 or 2 hundredths of the exact values.

Equations for the graphs

Equation for line #1: $t_2 = \frac{1}{3}D + \frac{1}{5}$

Equation for line #2: $t_2 = \frac{1}{7}D + \frac{1}{2}$

Students are likely to use the slope-intercept form (y = mx + b) of the equation of a line to find these formulas. However, they may also make tables and use patterns to find their answers. Encourage students to share and compare their different approaches.

Calculating the point of intersection

$$t_1 = t_2$$
$$\frac{1}{3}D + \frac{1}{5} = \frac{1}{7}D + \frac{1}{2}$$
$$\frac{1}{3}D - \frac{1}{7}D = \frac{1}{2} - \frac{1}{5}$$
$$\frac{7}{21}D - \frac{3}{21}D = \frac{5}{10} - \frac{2}{10}$$
$$\frac{4}{21}D = \frac{3}{10}$$
$$D = \frac{3}{10} \cdot \frac{21}{4} = \frac{63}{40} = 1.575$$

$$t_1 = \frac{1}{3} \cdot D + \frac{1}{5}$$
$$t_1 = \frac{1}{3} \cdot \frac{63}{40} + \frac{1}{5}$$
$$= 1 \cdot \frac{21}{40} + \frac{1}{5}$$
$$= \frac{21}{40} + \frac{8}{40} = \frac{29}{40} = 0.725$$

Both values are consistent with the estimate above.

Students may have varying approaches. For example, they may multiply through by the LCM of the denominators in order to eliminate fractions, or they may use a decimal form for some of the fractions. They may also check their value for t_1 by substituting 1.575 into the equation for t_2 and seeing that they obtain the same value, 0.725.

The meaning of the intersection point

For each girl, it takes 0.725 hours (43.5 minutes) of total exercise time (including stretching) to travel 1.575 miles. This is the one point at which it takes them the same amount of exercise time to travel the same distance.

Stage 2

In Stage 2, students build on their work in Stage 1 as they dig deeply into concepts related to linear inequalities.

In Problem #3, they extend their interpretation of their real-world context from Stage 1 by creating an algebraic inequality from a pair of graphs and analyzing connections between algebraic solution processes and the graph.

In Problem #4, students generalize their solution by allowing the quantities in the original situation to take on any meaningful values—that is, by representing all of the quantities as variables.

Finally, in Problem #5, students think in reverse. They are given a solution set and asked to create and analyze linear inequalities that have that solution set.

One of the best things that students can do to deepen and broaden their understanding of all of these concepts is to share and compare the contexts that they create, their methods for solving inequalities, and the reasoning behind their various approaches.

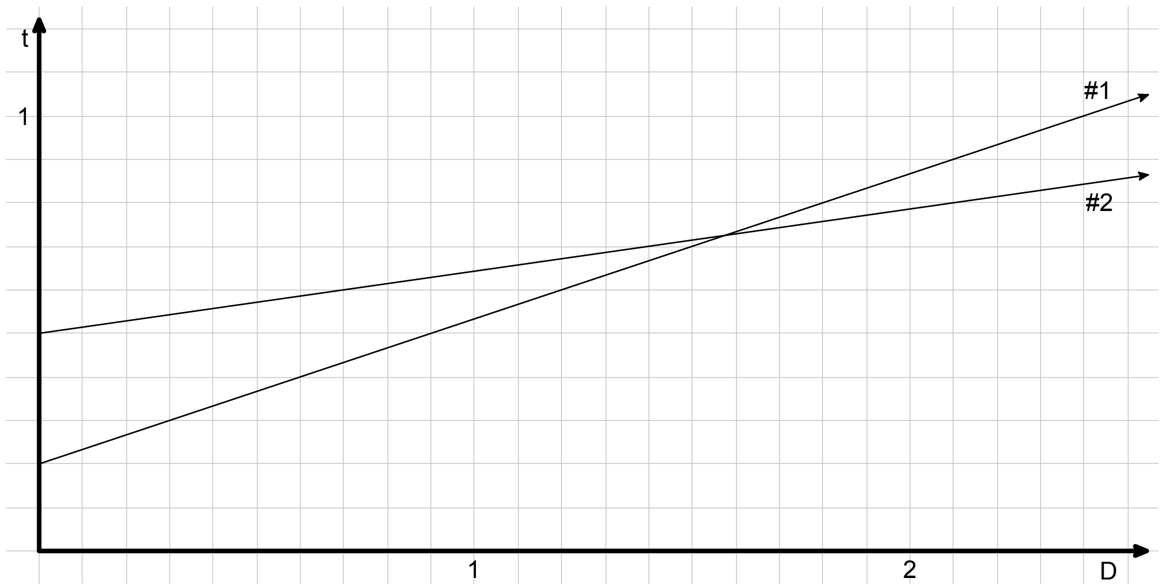
What students should know:

- Understand concepts from Stage 1.
- Know the meanings of the inequality symbols <, >, ≤, and ≥.
- Solve linear inequalities algebraically.

What students will learn:

- Deepen understanding of inequalities by analyzing relationships between inequalities and graphs, especially in the process of solving the inequalities.
- Make sense of the rule for "changing the direction" of the inequality symbol by analyzing a graph in a real-world context.
- Develop and apply methods for solving linear inequalities when the coefficients are expressed as variable parameters.
- Deepen understanding of inequalities by exploring relationships between terms coefficients, and solutions.

Problem #3



Directions

- Write an algebraic inequality for the situation $t_2 < t_1$.
- Solve the inequality.
- Explain what the inequality and solution mean in your story.
- Explain how the graphs show the solution to the inequality.

Conversation Starters for #3

What do you notice? What do you wonder?

I notice that my process for solving the inequality is almost the same as it was for solving the equation in Problem #2.

After students have solved the inequality

I notice that the solution to the equation is the "boundary point" for the solution to the inequality.

I wonder what happens if I move the variable terms to the opposite side of the inequality when solving it.

I notice that the graphs "trade places" on opposite sides of the intersection point (the graph that was higher becomes lower).

Solutions for #3

An algebraic inequality for $t_2 < t_1$.

$$\tfrac{1}{7}D + \tfrac{1}{2} < \tfrac{1}{3}D + \tfrac{1}{5}$$

Solving the algebraic inequality

$$\tfrac{1}{7}D + \tfrac{1}{2} < \tfrac{1}{3}D + \tfrac{1}{5}$$
$$\tfrac{1}{7}D - \tfrac{1}{3}D < \tfrac{1}{5} - \tfrac{1}{2}$$
$$-\tfrac{4}{21}D < -\tfrac{3}{10}$$
$$D > -\tfrac{3}{10} \cdot -\tfrac{21}{4}$$
$$D > \tfrac{63}{40}$$

Notes: Some students may place the "D" terms on the right side of the inequality, obtaining $\frac{63}{40} < D$. This is equivalent to the solution above. Ask students to share and compare their strategies for solving the inequality.

The meaning of the inequality and the solution

The inequality and the solution mean that, when both girls travel the same distance, Itsuko spends less time exercising then Karen does for distances greater than 1.575 hours (1 hour and 34.5 minutes).

How the graphs show the solution to the inequality

Itsuko's graph lies below Karen's graph at all points to the right of *D* = 1.575.

Problem #4

D: input
t_1: output for graph #1
t_2: output for graph #2
t_{1i}: initial value of output for graph #1
t_{2i}: initial value of output for graph #2
R_1: reciprocal of slope for graph #1
R_2: reciprocal of slope for graph #2

Directions

- Rewrite your equations from Problem #2 using these variable names.
- Rewrite the inequality for $t_2 < t_1$ using these variable names.
- Solve the inequality.
- Interpret your solution to the inequality in terms of your story.

Conversation Starters for #4

What do you notice? What do you wonder?

I wonder what "initial value" means.

The initial value of a variable is its "starting" value—its value when the input (D, in this case) equals 0.

I wonder what is the purpose of replacing all of the numbers by variables.

I notice that by using only variables, I am answering an infinite number of questions (one for every possible value of the variables)!

This is called *generalizing* the solution—that is, making it apply to *genera*l values of the variables instead of limiting it to specific numbers.

I notice that my inequalities are more challenging to solve when they contain variables instead of numbers.

I notice that the process of solving the inequality begins in the same way as it did in Problem #3.

This fact may help make the process feel more understandable and easier to start.

I notice that the process of combining like terms looks like factoring (using the distributive property) when the quantities are variables.

I wonder how to decide whether or not to change the "<" symbol to ">."

Deciding may seem impossible at first, since you can't tell whether the coefficient of D is positive or negative. In fact, the solution process splits into two separate cases, one for each possibility.

I notice that whether the coefficient of D is positive or negative depends on which girl is walking or jogging faster.

I wonder what happens to the inequality (or equation) if both girls move at the same speed.

Solutions for #4

Rewriting the equations

$$t_1 = \frac{1}{R_1} D + t_{1i} \qquad t_2 = \frac{1}{R_2} D + t_{2i}$$

Rewriting the inequality

$$\frac{1}{R_2} D + t_{2i} < \frac{1}{R_1} D + t_{1i}$$

Solving the inequality

$$\frac{1}{R_2} D - \frac{1}{R_1} D < t_{1i} - t_{2i}$$

$$\left(\frac{1}{R_2} - \frac{1}{R_1} \right) D < t_{1i} - t_{2i}$$

The next step depends on the relationship between R_1 and R_2.

If $R_1 < R_2$, then the reciprocal of R_1 is greater than the reciprocal of R_2. Therefore,

$$\frac{1}{R_2} - \frac{1}{R_1} < 0$$

Since the coefficient of D is negative, the relationship between the two sides changes from < to > when dividing both sides by it:

$$D > \frac{t_{1i} - t_{2i}}{\frac{1}{R_2} - \frac{1}{R_1}}$$

If $R_1 > R_2$, then the reciprocal of R_1 is less than the reciprocal of R_2. In this case,

$$\frac{1}{R_2} - \frac{1}{R_1} > 0$$

Since the coefficient of D is positive, the relationship between the two sides stays the same ("less than") when dividing both sides:

$$D < \frac{t_{1i} - t_{2i}}{\frac{1}{R_2} - \frac{1}{R_1}}$$

Note: You may rewrite the expression $\frac{t_{1i}-t_{2i}}{\frac{1}{R_2}-\frac{1}{R_1}}$ in the form $\frac{R_1R_2}{R_1-R_2}\left(t_{1i}-t_{2i}\right)$.

Interpreting the solution

The solution(s) will now work for any values of R_1, R_2, t_{1i}, and t_{2i} that make sense in the problem situation.

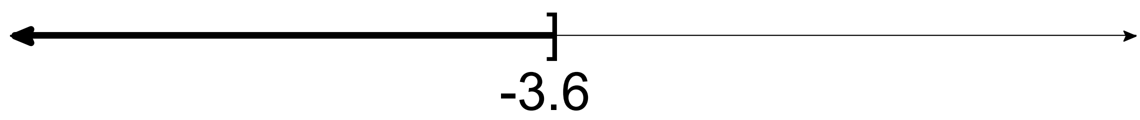
If $R_1 < R_2$, then Karen is the slower walker or runner (as in the original problem). In this case, the time required for her to travel farther than 1.575 miles is always *greater* than Itsuko's.

If $R_1 > R_2$, then Karen is the faster walker or runner. In this case, the time required for her to travel farther than 1.575 miles is always less than Itsuko's.

Students may also verify that the expression produces the correct result for the original boundary value.

$$\frac{R_1R_2}{R_1-R_2}\left(t_{1i}-t_{2i}\right)$$
$$=\frac{3\cdot 7}{3-7}\left(\frac{1}{5}-\frac{1}{2}\right)$$
$$-\frac{21}{4}\left(-\frac{3}{10}\right)=\frac{63}{40}$$

Problem #5



Directions

- Create at least three different inequalities in the form ax + b ≥ cx + d that have the solution set illustrated on the number line. Explain your thinking.
- Compare and contrast your inequalities.
- Invent and describe a general procedure for creating linear inequalities in the form ax + b ≥ cx + d that have a given solution set, x ≤ n.

Conversation Starters for #5

What do you notice? What do you wonder?

I wonder what the bracket means.

The bracket shows that the boundary number, -3.6, is part of the solution.

I wonder if it would help to look at my method for solving inequalities.

I notice that thinking backwards from the solution set to the inequality might be helpful.

I notice that the solution comes from dividing coefficients (after combining like terms).

I notice that combining like terms involves addition and subtraction.

I notice that the direction of the inequality symbol must change between the inequality and its solution.

I wonder if would help to create an algebraic expression (using a, b, c, and d) that calculates the boundary point of the solution.

Some students may be able to discover and make use of the expression $\frac{d-b}{a-c}$ for the boundary point.

Solutions for #5

Sample inequalities in the form ax + b ≥ cx + d

$6D + 2.1 \geq 9D + 12.9$
$7D + 8.1 \geq 10D + 18.9$
$12D + 10 \geq 14.5D + 19$

There are *many* other inequalities with the same solution set. Students may have a variety of strategies, including using estimation to guess values for the coefficients and then adjusting them until they get the desired result.

A systematic approach might involve thinking in reverse about equivalent equations and the solution process:

- Find two numbers that have a quotient of 3.6: for example, 10.8 ÷ 3
- Choose *a* and *c* so that their difference equals the divisor (3 in this case).
- Chose *c* to be greater than *a* so that the coefficient of *x* is negative. This ensures that the direction of the inequality symbol will need to be reversed.
- Choose *b* and *d* so that their difference equals the dividend (10.8 in this case).
- Choose *d* greater than *b* so that quotient (d – b)/(a – c) is *negative.*

Note: The first two sample inequalities use the quotient 10.8 ÷ 3. The second inequality preserves the differences *a – c* and *d – b* by adding the same number to each coefficient within each pair. The third inequality uses the quotient 9 ÷ 2.5.

Comparing and contrasting the inequalities

Whether they used the strategy above or not, students may note the relationships between *a, b, c,* and *d* described above.

A general procedure for creating linear inequalities having a given solution x ≤ n.

The process described above is a good procedure for creating linear inequalities having a given solution.

- Find two numbers whose quotient is *n* (or its absolute value).
- Choose *a* and *c* so that their difference equals the divisor.
- Choose *b* and *d* so that their difference equals the dividend.
- If the relational symbol on the original inequality is ≥, then choose *c* > *a*. Otherwise, choose *c* < *a*.
- To determine which of *b* or *d* is greater, make *d – b* have the same sign as *a – c* if *n* is positive. Otherwise, choose them so that *d – b* has the opposite sign as *a – c*.

Stage 3

Stage 3 contains one problem. Students are asked to look at a number line graph unlike any that they have seen before and create an algebraic inequality that has the given solution set. The key difference between this problem and Problem #5 is that the inequality cannot be linear.

This problem requires a lot of creativity, and it is likely that students will not find a solution on their own. However, the process of *trying* to find one will expand their ideas of what an algebraic inequality is and what a solution set can look like. It will also engage students in thinking carefully about the very meaning of the phrase *solution set*, and it will help them learn to think flexibly and to learn from efforts that fail.

The Conversation Starters may give you some ideas for helping students make progress in their thinking without give the answer away too soon. Eventually, by suggesting that they think about reciprocals, you may be able to lead them to an answer. As a side-benefit, they will gain a deeper understanding of what reciprocals mean and how they work.

Once students do find an answer (or you eventually give it to them), consider asking them to trying solving the inequality using the familiar methods for linear inequalities. They will discover that this is not possible, and they may become curious how they *could* go about solving an inequality like this. They will learn methods in later algebra courses, but if they do some of their own experimentation, they may be able discover some valuable methods themselves!

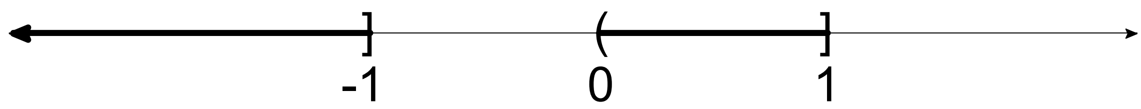
What students should know:

- Understand the concepts from Stages 1 and 2.

What students will learn:

- Gain a deeper understanding of the meaning of the term *solution set.*
- Recognize the existence of non-linear inequalities, and notice that their solution sets may have different features than those of linear inequalities.
- Think flexibly and persist in solving problems that have very little structure.
- Gain a deeper understanding of reciprocals.

Problem #6



Directions

Create a single algebraic inequality that has the solution set shown on the number line graph. Explain your thinking.

Diving Deeper

Create your own number line graph like the one above. Find an algebraic inequality that has the solution set shown on your graph. Can your inequality be solved algebraically using the same methods that you used earlier in the exploration?

Conversation Starters for #6

What do you notice? What do you wonder?

I wonder what the brackets and parentheses mean.

The brackets indicate boundary numbers that are included in the solution set. Parentheses indicate boundary numbers that are not.

I wonder why 0 would not be a solution when every other number from 0 to 1 is.

I wonder if it would be easier to start by ignoring the negative solutions and trying to find an expression whose positive solutions are the numbers between 0 and 1.

I wonder if it is possible to have a linear inequality with this solution.

I wonder what kinds of operations and expressions appear in equations (or inequalities) that are not linear?

You might suggest powers of variables and fractions that include variables in the denominators as possibilities.

Solutions for #6

An inequality that has the solution set shown in the graph

$$\frac{1}{x} \geq x$$

Although the inequality looks simple, students may find it very challenging to discover. After allowing them to explore for quite a while, you may hint that the inequality is not linear. Eventually, you may even ask them to think about reciprocals.

Another option (again, after students have explored for quite a while) is to give them the solution, have them test many values of *x*, and explain *why* the given graph shows the correct solution set. They should account for all regions of the graph as well as all three endpoints—especially for the fact that 0 is not a solution (They may say that it makes no sense to divide by 0, or equivalently, that 0 has no reciprocal).