Deep Algebra Projects: Pre-Algebra/Algebra 1
Triangle Area Patterns

Topics

- Areas of triangles (on a grid and with a formula)
- Slopes of lines; slopes of parallel and perpendicular lines; y-intercepts
- Equations of lines
- Solving linear equations (Stages 2 and 3)
- Finding intersection points of lines (Stages 2 and 3)
- Pythagorean theorem and distances in the coordinate plane (Stage 3)
- Radicals—properties and calculations (Stage 3)

In the Triangle Area Patterns activity, students begin with two vertices and search for the locations of a third vertex that will form a triangle having a specified area. The activity blends geometric measurement concepts with many ideas related to linear functions: slope of a line, y-intercepts, slope-intercept form, point-slope form, slopes of parallel and perpendicular lines, and systems of linear equations. All of this is done within a context of exploration and problem-solving.

Before you begin, consider your learning goals carefully, because many of the problems can be used in one of two ways: (1) as a means of *teaching* the concepts to students who have not yet been exposed to them, or (2) as a means of learning to *apply* the concepts for those who have already been taught them. The activity contains guidance about this issue throughout—in the Introductions to the stages, in the Conversation Starters, and in the Solutions.

Stage 1

Important note: Wait until students have completed each problem in Stage 1 before handing out the next one.

Problem #1 begins the activity with a focus on geometry—creating triangles with a given area. Students think and visualize flexibly as they search for multiple possibilities.

Problem #2 introduces a rectangular grid so that students can use coordinates to analyze the problem situation more deeply. They explore patterns involving coordinates and slope.

Students may attempt Problem #3 with or without previous knowledge of the slope-intercept form of linear equations. If they have had experience with slope-intercept form, the problem will give them an opportunity to apply their knowledge. If not, you may use the problem to guide students toward discovering some of the key ideas.

An optional but fun way to start Triangle Area Patterns is to act out a version of Problem #1. Ask two students to stand in different places in the room. Then name a realistic area, and ask another student to estimate a third location to stand in order to make a triangle whose area is close to the number that you named. Have the students make some measurements and do some calculations to see how close they were. Afterwards, try new locations of the third vertex in order to improve the results.

What students should know:
- Find areas of triangles.
- Name coordinates and plot points in a four-quadrant coordinate grid.
- Understand that the graph of an equation consists of all points whose coordinates make the equation true.
- Understand and calculate slopes of lines (recommended).
- Understand slope-intercept form for the equation of a line (optional).

What students will learn:
- Visualize and think flexibly about areas of triangles.
- Recognize and extend linear patterns visually and numerically.
- Use slope to describe patterns in points (and their coordinates) on lines.
- Find equations that describe linear patterns.
- Apply slope-intercept form for the equation of a line (if they already know it).

Problem #1
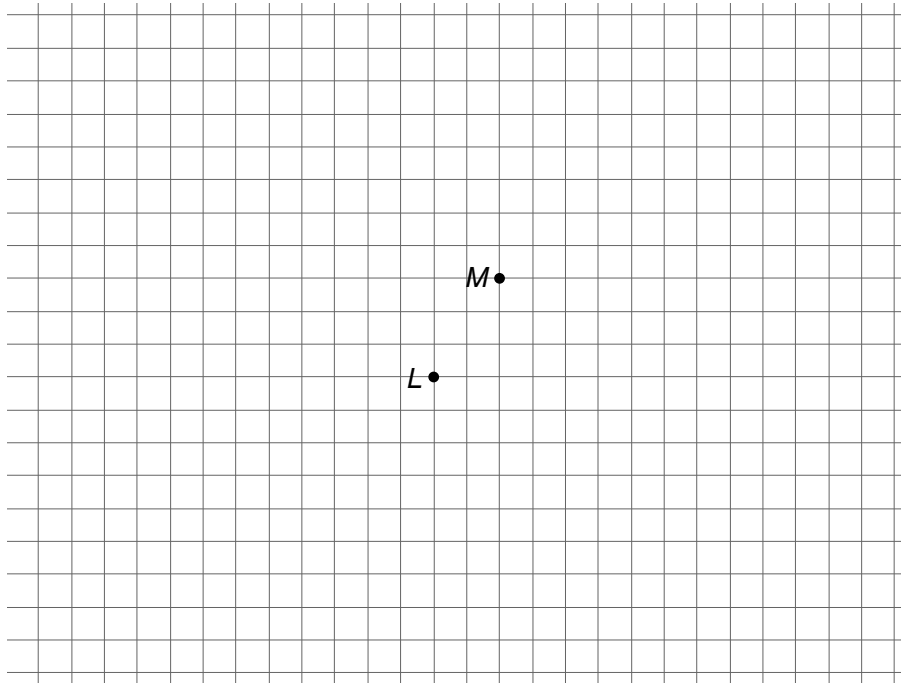
The area of ΔLMN is 7 units^2.

Directions

- Find a possible location for the vertex, N. Explain your thinking.
- If possible, show three more places where N could be. If it is not possible, explain why. Discuss any new ideas or strategies that you use.

Conversation Starters for Problem #1
What do you notice? What do you wonder?

I wonder what area each small square on the grid represents?
> For this problem, let's make the natural assumption that a small square has an area of 1 square unit. It might be interesting to see how it would affect the answers to the problems in this activity if you made a different choice!

I notice that if $\overline{LN}$ is horizontal, then the height is 3 units.

I notice that if $\overline{LN}$ is vertical, then the height is 2 units.

I wonder if $\overline{LN}$ has to be vertical or horizontal.

I wonder if N has to be at a point where the gridlines meet.

I wonder what strategies I could use to find the area if none of the sides were horizontal or vertical.

I wonder if I can decompose my triangle into shapes whose areas are easier to find.

I wonder what happens if I surround my triangle with a rectangle.

I notice that a triangle is always half of a parallelogram.

I wonder if it would help to begin by imagining parallelograms on the grid?

I wonder what happens to the area when I move N up or down 1 unit.

I notice symmetry in some of the possible locations for N.

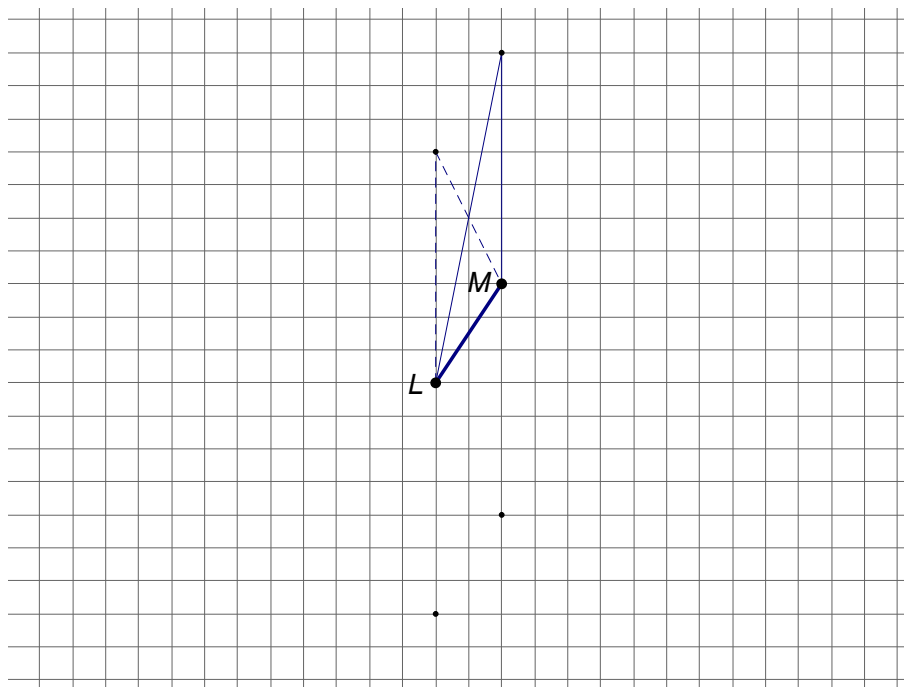*I wonder* how many locations for N are possible.

I wonder if there are patterns in the possible locations for N.

I wonder what this problem has to do with algebra.

Solutions for #1

Sample locations for N

One approach is to search for base/height combinations that give an area of 7 units^2. This occurs whenever $bh = 14$.

Four possible locations (small dots) are shown above. They are 7 units above and below points L and M, respectively. The triangles LMN that are formed each have a base of 7 units and a height of 2 units. Two of these triangles are illustrated.

Some students may notice parallelograms containing $\overline{LM}$ formed by these points. Each triangle is half of one of these parallelograms.

Another interesting observation:
Each time you move the point N up or down 1 unit (directly above or below L or M), the area changes by 1 unit^2. Can you see why this happens? (It is related to the fact that the height is 2 units.)

More possible locations for N appear on the following pages. For a more complete list, see the Solutions for Problem #2.

You may use a similar strategy thinking horizontally.
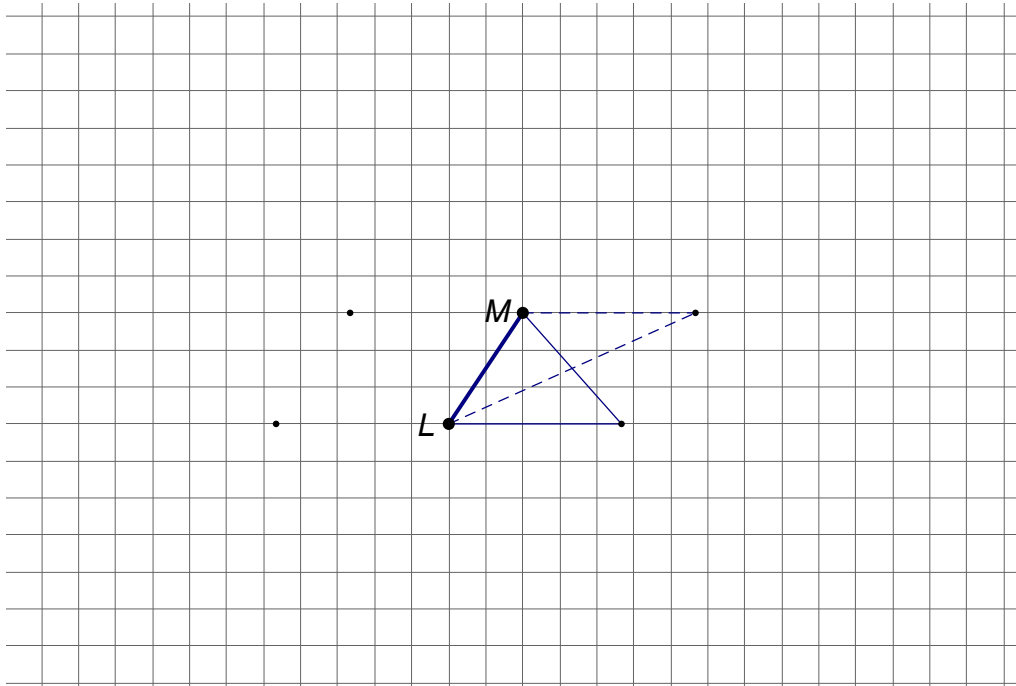
Each of the four locations shown here is $4\frac{2}{3}$ units to the left or right of L or M. This results in triangles having bases of $4\frac{2}{3}$ units and heights of 3 units. Two of these triangles are illustrated. Again, students may notice that the dots form parallelograms containing $\overline{LM}$ and that each triangle is half of such a parallelogram.

Whenever you move the point N left or right 1 unit along a horizontal line through L or M, the area of the triangle changes by 1.5 unit². Can you see why this happens? (It is related to the fact that the height is 3 units.)

Another approach is "surround and subtract"—that is, select locations for N by estimation and then use the grid directly (without a formula) to determine the area. For example:
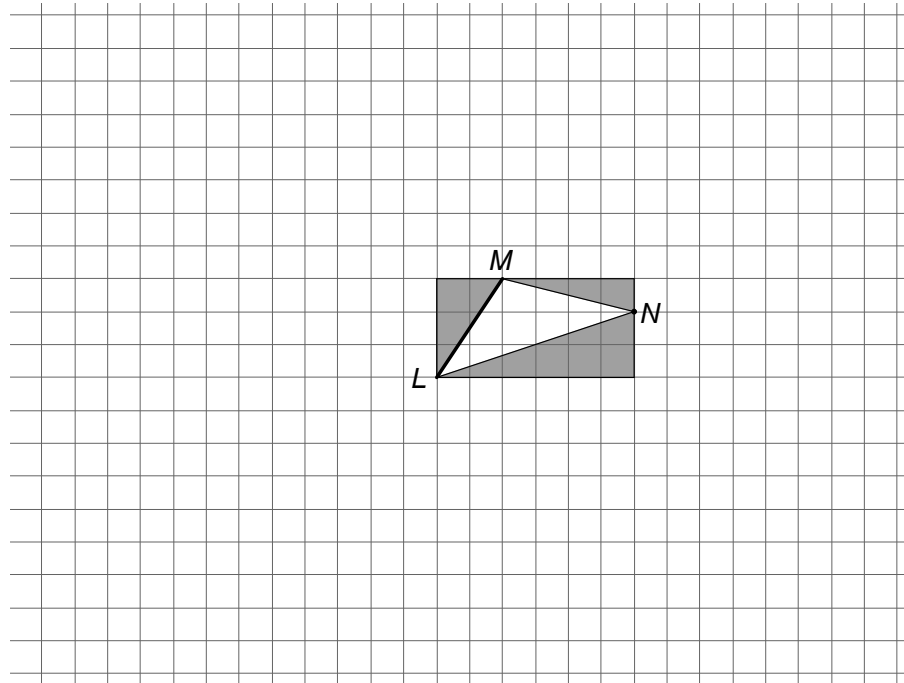

The illustrated location of N is 4 units to the right of and 1 unit down from M. One strategy for finding the area of ΔLMN is to find the area of the surrounding rectangle and then subtract the areas of the shaded triangles (which are easier to determine, because they are *right* triangles).

Rectangle – upper left – upper right – lower:
$$18 - 3 - 2 - 6 = 7$$

Other strategies and locations for N are possible. See Problem #2 for a more complete list of locations for N.

Problem #2
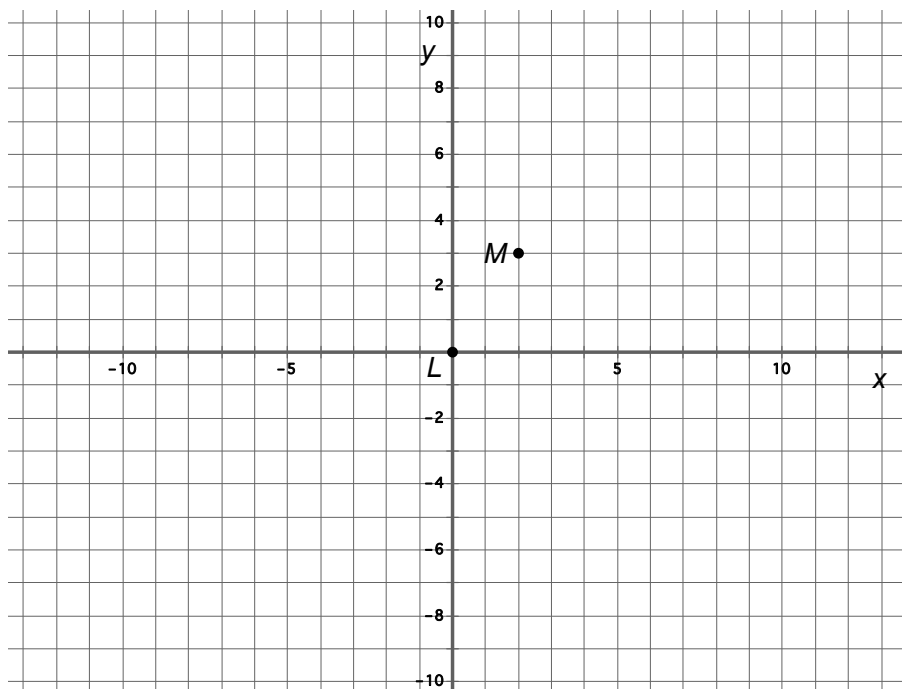
The area of ΔLMN is 7 units^2.

Directions

- Write the coordinates of the locations you have discovered so far for N.
- Show all possible *lattice point** locations for N on this grid. Find their coordinates.
- Describe any patterns that you have noticed, and explain what causes them.
- Show how to use the patterns to find a possible *non*-lattice point for N. Explain your thinking.

*A *lattice point* is a point where the two grid lines intersect. In other words, both coordinates of a lattice point are integers.

Conversation Starters for Problem #2
What do you notice? What do you wonder?

I notice that L and M are in the same relative positions as they were in Problem #1.

I notice that L is at the origin. *I wonder* if that will make the problem easier.

I notice that some of my answers from Problem #1 are already lattice points.
    This will almost certainly be the case for most students.

I wonder how I can find enough answers in order to begin noticing patterns.
    You might want to consider sharing answers with other students.

I notice that a lot of the points seem to be lining up.

I notice two sets of points, each of which makes its own line.

I notice that I can find a new point on either line by starting at a point that I know and moving right 2 and up 3 (or left 2 and down 3).

I wonder why this process always creates a new triangle with the same area.

I notice that both lines are parallel.

I wonder what the slopes of these lines are.

I notice that some of the points have opposite x- and y-coordinates.

I wonder what happens if I think of all of the triangles as having the same base, $\overline{LM}$.

I wonder what the common height of all of these triangles is.

I wonder what strategies I can use to find coordinates of points between two lattice points.
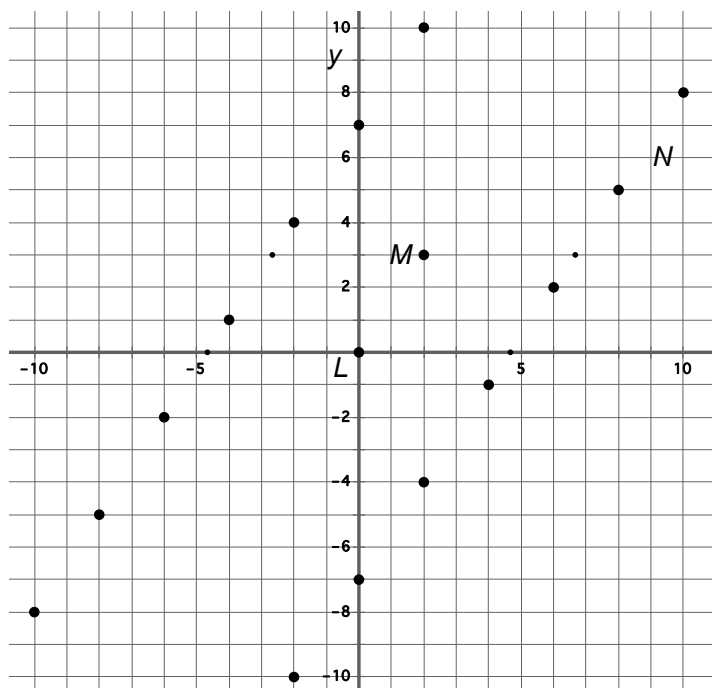
Solutions for #2

Coordinates of points discovered so far

Students should have a list of at least four coordinate pairs. Their answers will vary depending on their results from Problem #1. Most of their answers are likely to be included in the list below.

All lattice points for N on the grid

| (-2, -10) | (0, -7) | (2, -4) | (4, -1) | (6, 2) | (8, 5) | (10, 8) |
| (-10, -8) | (-8, -5) | (-6, -2) | (-4, 1) | (-2, 4) | (0, 7) | (2, 10) |

Notes: (1) Students are likely to search for a pattern (see the next page) as a *means* of discovering all of the lattice points. In order to do this, they may first need to search for more potential locations of N or share points discovered between themselves. (2) The small dots are not part of the answer to this question, because they show the four *non*-lattice points mentioned in the solution to Problem #1. If students have already discovered some of these points, you might consider asking them to include the points in their graphs anyway (in order to suggest the possibility that *all* points on the lines may create equal-area triangles).

Patterns

The points appear to fall on two lines, one of either side of $\overline{LM}$. Detailed observations may be numeric or visual.

Numeric Patterns

It appears that whenever (a, b) is a possible location for N, $(-a, -b)$ is as well. (These pairs of points are symmetric across the origin.)

There are two sets of coordinates, one containing (0, 7) and the other containing (0, -7). Both sets show the same pattern of change. Beginning at any known lattice point, continually increase (or decrease) the x-coordinate by 2 and y-coordinate by 3 in order to obtain a "neighboring" lattice point.

Visual patterns

All of the points fall on one of two lines, both of which are parallel to $\overleftrightarrow{LM}$. Both lines have a slope of $\frac{3}{2}$, which is the same as the slope of $\overleftrightarrow{LM}$, and both are the same distance from $\overleftrightarrow{LM}$. (It is not easy to see at first what this distance is, because distances between lines are measured perpendicular to the lines. Curious students will explore this question later in the project.)

Students can make (or reinforce) an important connection here: Lines are parallel if and only if they have equal slopes.

Causes of the patterns

Think of $\overline{LM}$ as the base of each possible triangle. In order for the area to remain the same, you must choose N so that the height of the triangle stays the same.
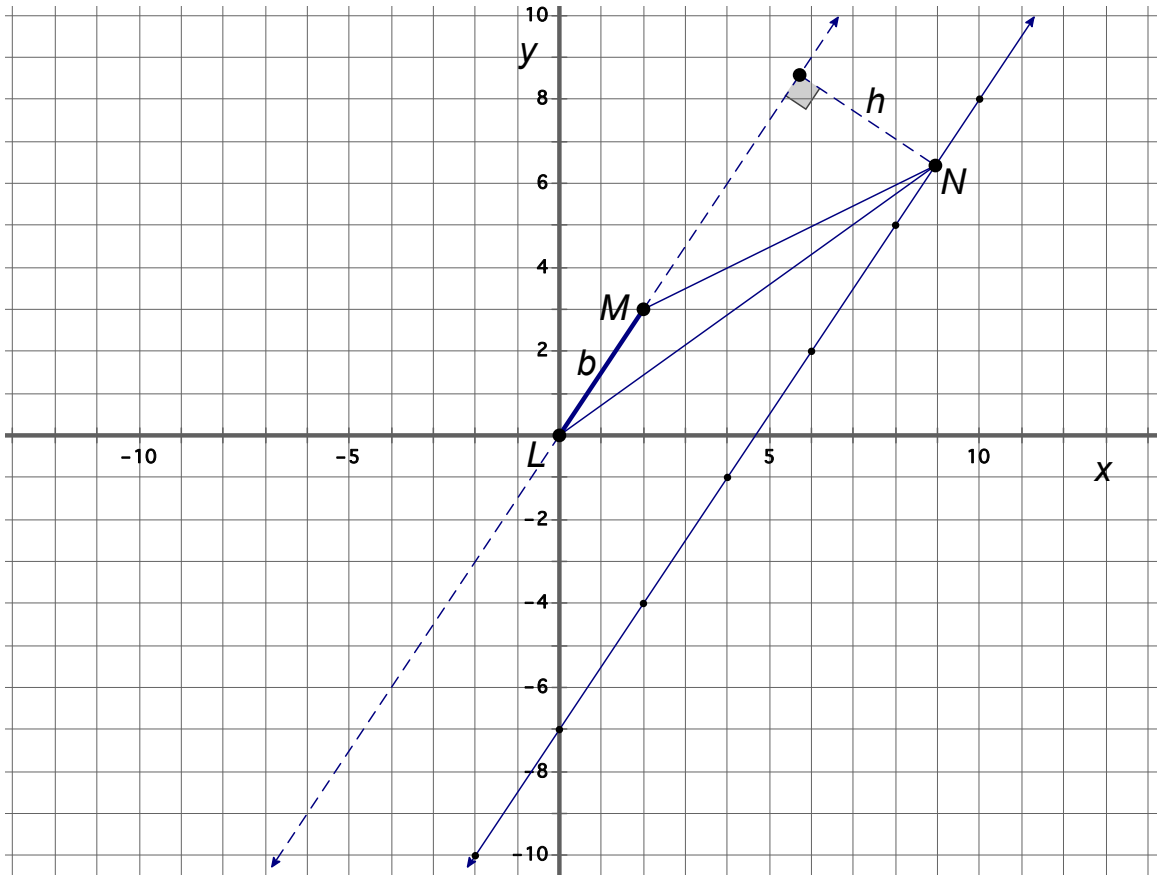
Imagine sliding N along the solid line. Because this line is parallel to $\overline{LM}$, the height of the triangle—and thus its area—never changes! Of course, you can do the same thing with the line on the opposite side of $\overline{LM}$.

Finding a non-lattice point for N

There are many approaches. If you have an equation for one of the lines (which students will find in Problem #3), you could choose an x or y value and then solve for the other variable. Without the equation, you could (for example) find the midpoint between two consecutive lattice points. For instance, the midpoint between (6, 2) and (8, 5) is (7, 3.5). Since this midpoint lies on the same line as the chosen lattice points, it will also create a triangle having an area of 7 units^2. (Students may verify the area by drawing a picture like the one on the final page of the Problem #1 Solutions.)

Problem #3

In Problem #2, you discovered that the possible locations for the third vertex of the triangle lie along two lines.

Directions

- Find an equation for each line. Explain your thinking.
- Find an equation for the line $\overleftrightarrow{LM}$. Explain your thinking.
- Compare and contrast all three equations.
- Test your equations with coordinates for some possible locations of N.

Conversation Starters for Problem #3
What do you notice? What do you wonder?

Note: Students' thinking on this question will depend on whether they have already learned about the slope-intercept form of the equation of a line. Many of the items below apply best to students who have not yet learned it and are thus developing their *own* strategies. Some who *are* familiar with slope-intercept form may solve Problem #3 fairly quickly and be ready to take on the challenges of Stage 2.

I notice that both lines have the same slope.

I notice that this slope is the same as the slope of the line through L and M.

I notice that the slope determines how fast the x- and y- coordinates change together.

I wonder if knowing the slope will help me find an equation.

I wonder if it would be easier to find the equation of the line $\overleftrightarrow{LM}$ *first*.

I notice connections between the visual characteristics of the graphs and the numbers in the equations.

Solutions for #3

Equations for the lines containing the third vertex

An equation for the upper line is $y = \frac{3}{2}x + 7$.

An equation for the lower line is $y = \frac{3}{2}x - 7$.

Students who have learned about the *slope-intercept* form of the equation of a line may find these equations fairly quickly. Others may need to examine tables of x- and y-coordinates and play around with them until they find equations that work. After they have finished, they could analyze their results and recognize that the equations "happen to" contain the slope and y-intercepts of the lines!

An equation for the line $\overleftrightarrow{LM}$

Since its slope is $\frac{3}{2}$, and its y-intercept is 0, the equation of the line is $y = \frac{3}{2}x$.

Comparing and contrasting the equations

All three equations have an x-coefficient of $\frac{3}{2}$, reflecting the fact that all three lines are parallel and have a slope of $\frac{3}{2}$.

The three y-intercepts are -7, 0, and 7. They are equally spaced. Some students may wonder if the y-intercepts will always be equal to or the opposite of the area of the triangles. (This would be an excellent question to explore. It happened in this case only because the heights of the triangles with vertical bases are always equal to 2—in other words, because the x-coordinate of M is 2 greater than the x-coordinate of L).

Examples of testing the equations

Coordinates: (-2, -10)   Equation: $y = \frac{3}{2}x - 7$

$$y = \frac{3}{2}x - 7$$
$$\frac{3}{2}(-2) - 7 = -3 - 7 = -10 \quad \checkmark$$

Coordinates: (-8, -5)   Equation: $y = \frac{3}{2}x + 7$

$$y = \frac{3}{2}x + 7$$
$$\frac{3}{2}(-8) + 7 = -12 + 7 = -5 \quad \checkmark$$

Stage 2

Stage 2 consists of two problems in which students generalize their formulas from Problem #3, find equations for lines whose y-intercepts may not be quickly apparent, and find coordinates for points where lines intersect. Both problems take place in the context of exploring equal-area triangles in coordinate grids.

Items under "What students should know" are listed as *recommended* if students are more likely to have success with the problems and to learn the most from them if they have the specified knowledge or experience. Students who do not have these prerequisites may still be able to solve the problems, but the work may be much more challenging, because they will need to develop strategies that extend their current knowledge by quite a bit.

Items are listed as *optional* if the problems are useful for students whether or not they have the specified knowledge or experience. If they already have the prerequisite, then they will need to apply their knowledge to the problem. Otherwise, they will need to use their conceptual understanding to develop their own strategies. This may be challenging, but they are likely to succeed with sufficient time, perseverance, and support.

What students should know
- Understand the problems in Stage 1.
- Understand the slope-intercept form of the equation of a line (recommended).
- Understand the point-slope form of the equation of a line (optional).
- Solve systems of linear equations (optional).

What students will learn
- Write linear equations in situations with general (non-numerical) coordinates.
- Test generalized equations against specific cases.
- Recognize intersection points of graphs as points that satisfy multiple conditions.
- Develop strategies for finding coordinates of intersection points of graphs.
- Solve systems of linear equations (if they have learned how) in order to solve problems.

Problem #4
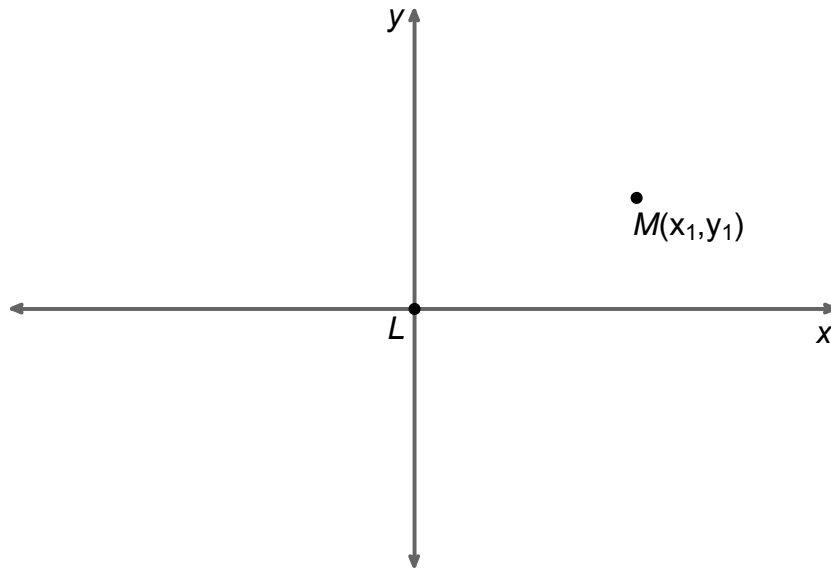
The area of ΔLMN is A units^2.

Directions

- Find equations for the lines containing all possible vertices for N.
- Verify that your general equations produce the particular equations in Problem #3 when the area is 7 units^2 and coordinates are (0,0) and (2, 3).

Diving Deeper

The point M is shown in the upper-right quadrant (Quadrant I). Will your equations change if M is in a different quadrant? If so, how will they change? If not, why not?

Conversation Starters for Problem #4
What do you notice? What do you wonder?

I notice that L is at the origin, and M is in the upper-right quadrant.

I wonder if my equations are going to have variables in the places where my equations from Problem #3 had numbers.

I wonder if my equations will have *any* numbers in them!

I notice that the slope of the lines is not a definite number, but its algebraic expression is not too hard to determine.

I wonder how I can find the y-intercepts of the lines.

I notice that it may help to *start* by looking for the points that lie on the y-axis.

I notice that the absolute value of the y-intercept is the length of a vertical base of a triangle.
    This happens because L is at the origin.

I wonder if I would get the same expressions for the y-intercepts no matter what points on the lines I found first.
    For example, the items above suggest starting by finding points on the y-axis. What if you began by finding points on the x-axis instead?

I wonder what happens if M is not in the upper-right quadrant.
    See the Diving Deeper question.

Solutions for #4

The equations of the lines

$$y = \frac{y_1}{x_1}x + \frac{2A}{x_1}$$

$$y = \frac{y_1}{x_1}x - \frac{2A}{x_1}$$

The thinking behind the equations

You may write the equations in the slope-intercept form, $y = mx + b$, where m is the slope, and b is the y-intercept.

Since the lines are parallel to the line, $\overleftrightarrow{LM}$, the slope, m, is the same as the slope of $\overleftrightarrow{LM}$, which is $\frac{y_1}{x_1}$.

You must select the y-intercepts so that the lines are at the distances from $\overleftrightarrow{LM}$ that produce the correct area. When N lies on the y-axis and you think of $\overline{LN}$ as the base of $\triangle LMN$, the height of the triangle is the x-coordinate of M—that is, x_1. Since $\frac{1}{2}bh = A$:

$$bh = 2A$$
$$bx_1 = 2A$$
$$b = \frac{2A}{x_1}$$

The variable, b, in these equations represents the length of the base of the triangle (not the b in the slope-intercept formula). However, since L is at the origin, b also stands for the distance of N from the origin, which turns out to be the common *absolute value* of the desired y-intercepts! Thus, the two y-intercepts lie at $\pm\frac{2A}{x_1}$ on the y-axis.
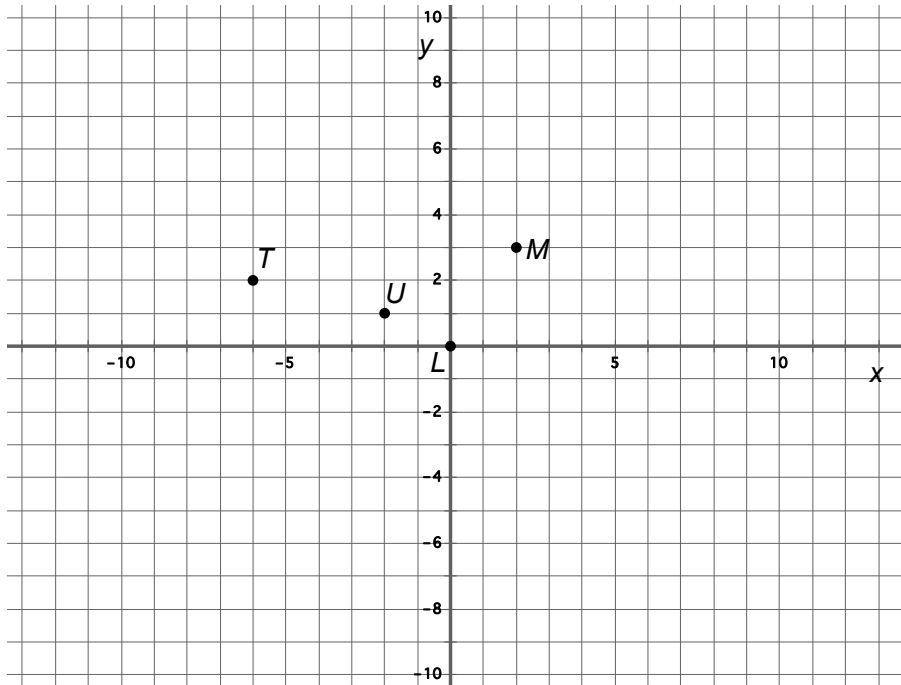
Verifying the equations

When $A = 7$ and $(x_1, y_1) = (2, 3)$,

$$y = \frac{y_1}{x_1}x + \frac{2A}{x_1} \text{ becomes } y = \frac{3}{2}x + \frac{2 \cdot 7}{2} = \frac{3}{2}x + 7. \quad ✓$$

$$y = \frac{y_1}{x_1}x - \frac{2A}{x_1} \text{ becomes } y = \frac{3}{2}x - \frac{2 \cdot 7}{2} = \frac{3}{2}x - 7. \quad ✓$$

Problem #5

The area of ΔTUV is 9 units^2.

Directions

- Graph the set of all possible points for V. (Ignore L and M for now.)
- Find the equation(s) of the graph(s).
- Determine the number of points, K, for which the area of ΔKLM is 7 units^2 (see Stage 1) *and* the area of ΔKTU is 9 units^2. Explain your thinking.
- Find (or estimate) the coordinates of at least one of these points.

Conversation Starters for Problem #5
What do you notice? What do you wonder?

I notice that the equations might be messy because neither T nor U is at the origin.

I wonder if would still help to search first for points on the y-axis.

I wonder if it would be easier to begin by searching for points vertically above or below T or U.

I notice that the two y-intercepts are no longer opposites of each other.

I notice that in order to find points satisfying both area conditions, it helps to look at all four lines at once.

    This observation refers to the lines from the first part of this problem and the ones from Problem #2.

I notice that graphs can help me predict the *number* of answers to a question before I even know the actual answers!

I notice that the lines do not intersect at lattice points, which makes the final question a lot harder.

I wonder how closely I can estimate the coordinates of the intersection points.

I wonder which of the four points would be the easiest to find exact coordinates for.

I notice that when two lines intersect at a certain point, it means that the x- and y- coordinates of the point satisfy *both* equations.
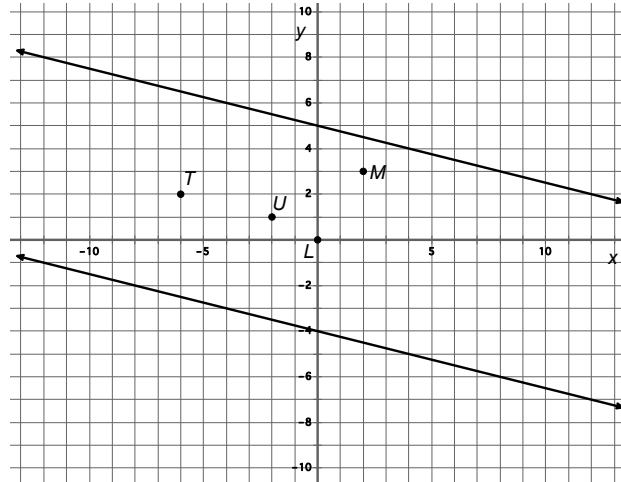
I wonder how I can express this idea algebraically.

I notice that I can apply my knowledge of solving linear equations.

I notice that I can look at my graph to check if my processes and answers make sense.

Solutions for #5

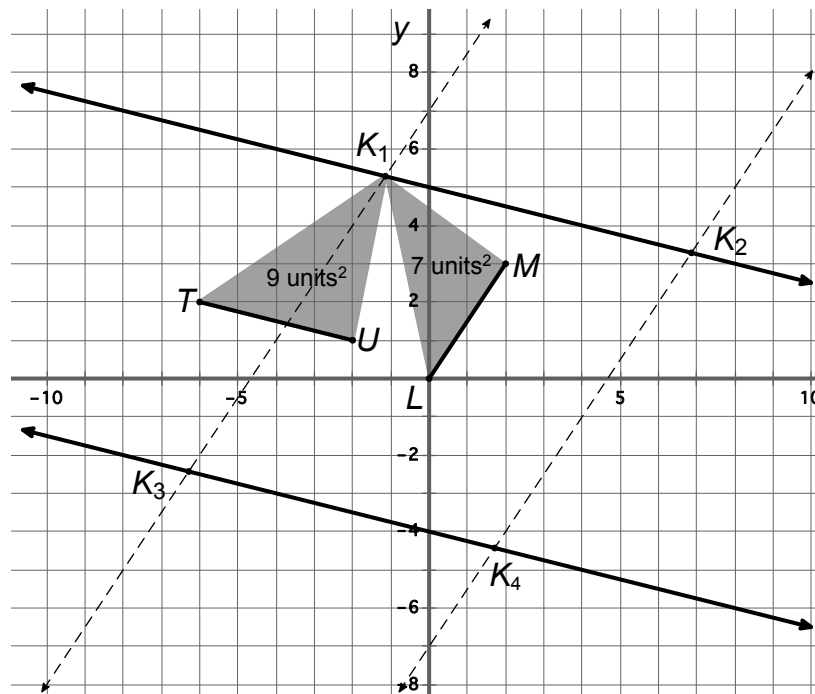*Graphs of the set of possible locations of V*

Equations of the graphs

$$y = -\frac{1}{4}x + 5 \text{ and } y = -\frac{1}{4}x - 4$$

The number of coordinates satisfying both area conditions

There are four such points (labeled $K_1, K_2, K_3,$ and K_4 in the picture below). They lie at the intersections of the lines from Problem #2 and the lines above.

The coordinates of the points

K_1: $\left(-\dfrac{8}{7}, \dfrac{37}{7}\right) \approx (-1.14, 5.29)$

K_2: $\left(\dfrac{48}{7}, \dfrac{23}{7}\right) \approx (6.86, 3.29)$

K_3: $\left(-\dfrac{44}{7}, -\dfrac{17}{7}\right) \approx (-6.29, -2.43)$

K_4: $\left(\dfrac{12}{7}, -\dfrac{31}{7}\right) \approx (1.71, -4.43)$

Note: Remember that students are asked to find or estimate *one* or more of these.

Possible strategies for finding the coordinates

Students' strategies may depend on whether they have already been taught algebraic procedures for finding coordinates of intersection points of graphs.

Some students may use the graphs to estimate the coordinates of an intersection point and then use a guess, test, and revise strategy to improve their estimates. With insight and lots of persistence, some may even be able to use proportional reasoning skills to pinpoint exact values.

Other students may be able to determine exact values by discovering or applying an algebraic procedure. They may reason that an intersection point occurs when the x- and y-coordinates on each graph, respectively, are equal. Therefore, it makes sense to set the expressions for two of the y values equal to each other and to solve the equation for x (in order to find the single value of x that makes them equal).

For example, in order to determine the coordinates of K_1, which lies at the intersection of the lines with the equations $y = -\frac{1}{4}x + 5$ and $y = \frac{3}{2}x + 7$, students may solve the equation:

$$-\frac{1}{4}x + 5 = \frac{3}{2}x + 7$$
$$-\frac{1}{4}x - 2 = \frac{3}{2}x$$
$$-2 = \frac{7}{4}x$$
$$-\frac{8}{7} = x$$

To find the corresponding y-coordinate, substitute this value into either equation.
$$y = \frac{3}{2}x + 7 = \frac{3}{2} \cdot -\frac{8}{7} + 7 = -\frac{12}{7} + \frac{49}{7} = \frac{37}{7}$$

Stage 3

In Problem #6, students try to find coordinates for vertices that result in *right* triangles. In the process, they either discover or apply knowledge of slopes of perpendicular lines.

In Problem #7, students work out the ultimate generalization of their formulas in a situation in which neither the area nor either set of coordinates is specified. The result is a complicated pair of equations which students may write in either point-slope or slope-intercept form. The latter is probably more challenging to produce but may be easier to work with once it is obtained.

What students should know
- Understand the ideas from Stages 1 and 2.
- Understand or be able to discover that slopes of perpendicular lines are opposite reciprocals.
- Understand and apply the Pythagorean theorem.
- Understand and apply properties of square roots.
- Understand and apply the slope-intercept form of the equation of a line.
- Understand and apply the point-slope form of the equation of a line (recommended).

What students will learn
- Discover or apply knowledge of slopes of perpendicular lines.
- Become more fluent in finding intersection points of lines.
- Apply knowledge of the Pythagorean theorem and square roots to calculate areas of triangles in a coordinate grid when no sides are horizontal or vertical.
- Apply knowledge of slope-intercept and/or point-slope forms of linear equations in situations in which numerical values of coordinates are not specified.
- Understand the relationship between general formulas and specific instances of them.

Problem #6

Certain locations for the vertex, N, (see Problem #1) result in *right* triangles.

Directions

- Referring to Problems #1 – 3, determine the number of locations for N that result in right triangles. Explain your thinking.
- Find the coordinates of at least two of these points.
- Verify independently that one of the resulting right triangles has an area of 7 units^2.

Diving Deeper

- Find the maximum area for which ΔLMN can have a right angle at N.
- Find the lengths of the sides of ΔLMN and coordinates for N in this case.

Conversation Starters for Problem #6
What do you notice? What do you wonder?

I notice that I can predict the number of possible locations for the points just by drawing a rough diagram.

I notice that it is not too hard to estimate the locations of N that cause the right angle in the triangle to be at L or M (if I draw a careful picture).

I wonder if it is possible for the right angle to be at N.

I wonder if I can use the slope of a given line to predict the slope of a line perpendicular to it.

I notice that rotating $\overline{LM}$ by 90° (holding L fixed) might help me find the slope of the line that is perpendicular to it.

I notice that when I do this, M does not quite reach the line containing the Ns.

I wonder what the new equation of the line would be after I do this rotation.

I notice that after I find the coordinates of one location for N that makes a right triangle, it is easier to find coordinates of the other three locations.

I wonder what it means to verify something *independently*.
    In this case, it means to use a completely different method of doing the calculation. Verifying
    something independently provides strong evidence that your original answer was correct.

I notice that I can use the Pythagorean theorem to find bases and heights for the right triangles.
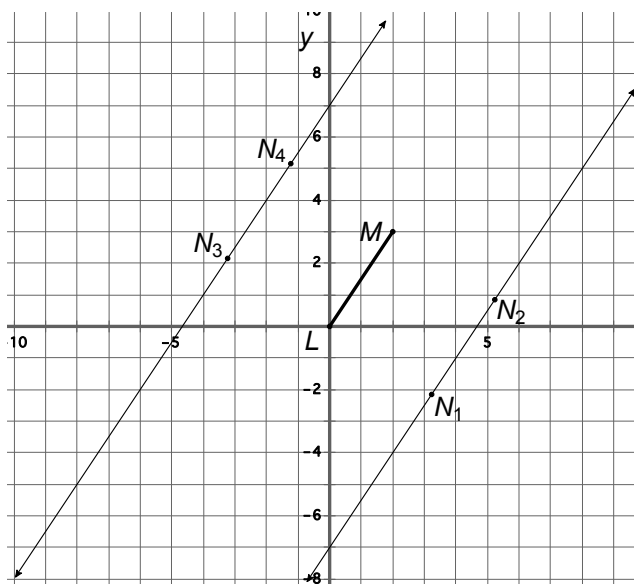
I wonder if there is a way to calculate the areas of the right triangles without using the Pythagorean theorem.
    Yes, there is. See the Solutions to Problem #1 and look at the third (surround and subtract) strategy.

Solutions for #6

The number of locations for N that result in right triangles

There are 4 possible locations for N (shown as $N_1, N_2, N_3,$ and N_4 below).

The easiest way for students to recognize that there are such four points is probably to draw a picture like the one above.

The right angles of the triangles will always lie at vertex L or M. Students should consider the possibility that the right angle could lie at a vertex N, but it is pretty apparent visually that the angles at N will always be acute for any point on either of the two lines. (If the area were smaller, the two lines would be closer to $\overline{LM}$ and it might be possible for this vertex to form a right angle. See the Diving Deeper question.)

The coordinates of the points

N_1: $\left(\dfrac{42}{13}, -\dfrac{28}{13}\right) \approx (3.23, -2.15)$

N_2: $\left(\dfrac{68}{13}, \dfrac{11}{13}\right) \approx (5.23, 0.85)$

N_3: $\left(-\dfrac{42}{13}, \dfrac{28}{13}\right) \approx (-3.23, 2.15)$

N_4: $\left(-\dfrac{16}{13}, \dfrac{67}{13}\right) \approx (-1.23, 5.15)$

Note: Remember that students are asked to find only *two* pairs of coordinates.

It is probably easiest to find the coordinates for N_1 (or N_3), because L is at the origin.

Experimenting visually or using their knowledge of slopes of perpendicular lines, students may discover that the slope of $\overleftrightarrow{LN_1}$ must be $-\frac{2}{3}$. (In particular, the point at (3, −2) must lie on this line.) Since this line passes through the origin, its equation is $y = -\frac{2}{3}x$. N_1 is the point at which this line intersects the line having the equation $y = \frac{3}{2}x - 7$.

$$-\frac{2}{3}x = \frac{3}{2}x - 7$$

$$-\frac{13}{6}x = -7$$

$$x = \frac{42}{13}$$

To find the corresponding value of y:

$$y = -\frac{2}{3}x = -\frac{2}{3} \cdot \frac{42}{13} = -\frac{28}{13}$$

Thus, the coordinates of N_1 are $\left(\frac{42}{13}, -\frac{28}{13}\right)$.

You may find coordinates of other points using simple transformations. For example:
- Find the coordinates of N_3 by reflecting N_1 over $\overline{LM}$.
- Find the coordinates of N_2 and N_4 by translating N_1 and N_3 respectively by (2, 3).

Independent verification that ΔLMN_1 has an area of 7 units^2

If $\overline{LM}$ is the base, then $\overline{LN_1}$ is the height. You may use the Pythagorean theorem with the shaded triangles below to find the lengths of these segments.
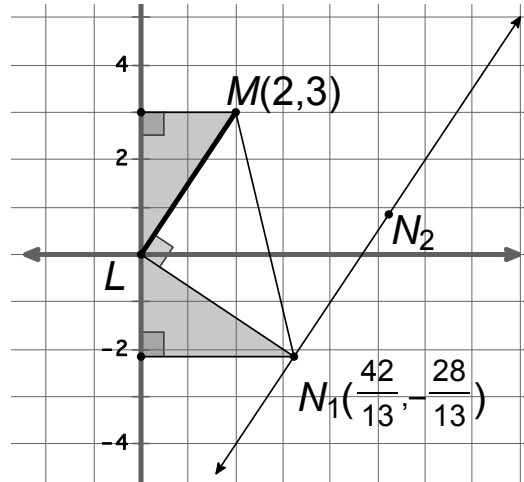
The length of the base, $\overline{LM}$, is

$$\sqrt{2^2 + 3^2} = \sqrt{13}.$$

The length of the altitude, $\overline{LN_1}$, is

$$\sqrt{\left(\frac{42}{13}\right)^2 + \left(\frac{28}{13}\right)^2} = \sqrt{\frac{1764}{169} + \frac{784}{169}} = \sqrt{\frac{2548}{169}} = \sqrt{\frac{196}{13}} = \frac{14}{\sqrt{13}}$$

The area of ΔLMN_1 is

$$A = \frac{1}{2}bh = \frac{1}{2}\sqrt{13} \cdot \frac{14}{\sqrt{13}} = \frac{1}{2} \cdot 14 = 7 \text{ units}^2 \checkmark$$

Note: Some students may verify the area without using the Pythagorean theorem by surrounding ΔLMN_1 with a rectangle, finding its area, and subtracting the areas of the right triangles that do not belong. This process will be at least as messy as the one above, because it will involve many mixed number computations.

Problem #7
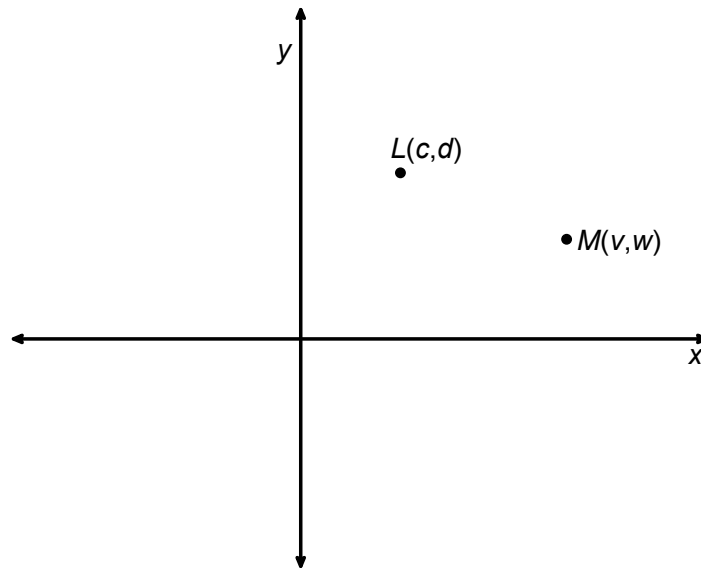
The area of ΔLMN is A units^2.

Directions

- Find equations for the lines containing all possible vertices for N.
- Figure out what happens to your equations if L is at the origin.
- Test your equations using the data from Problem #5.

Diving Deeper

Does your formula change if L and M have different relative positions or if they are in different quadrants? Explore!

Conversation Starters for Problem #7
What do you notice? What do you wonder?

I notice that no information about numbers is included in this question!

I wonder if my equations will have any numbers in them.

I wonder if I can use the same strategies for variables that I used for numbers.

I notice that it is easiest to use a horizontal or vertical base for my triangles.
> Yes—this makes it much easier to find an expression for the height! *Note:* The Solutions use a vertical base containing L.

I notice that certain parts of my expressions remind me of the equations I found in Problem #3.

I notice that some of the expressions get very complicated!

I wonder if there is a quick way to find an equation for a line if the point that I know is not the y-intercept.
> Students who know about point-slope form will have an easier time of it. Those who don't will have the excellent challenge of using their conceptual understanding to create their own strategies!

I wonder if it is important to write my final equations in slope-intercept form.
> It's okay to leave an equation in point-slope form, especially in a case like this when it would be a very messy job to rewrite it in slope-intercept form. (Also, many students may not yet have enough fluency with algebraic procedures to rewrite it—though it might be great practice to try!)

I notice that my two equations are nearly the same as each other.
> Yes—the only difference is in the sign of one of the terms!

I notice that once I substitute numbers for all of the coordinates and the area, the only variables left are x and y—just like usual for equations of lines!

I wonder if the equations would change if M were above and/or to the left of L.

I notice that I would get the same two equations if I exchanged the coordinates for L and M!

Solutions for #7

Equations of the lines

$$y = \frac{w-d}{v-c}x + \frac{dv - cw + 2A}{v-c}$$

$$y = \frac{w-d}{v-c}x + \frac{dv - cw - 2A}{v-c}$$

These equations are written in slope-intercept form. Students may write them in other ways. (See the next page.) Most students will begin by finding the slope of the lines, which is $\frac{w-d}{v-c}$. In order to progress further, they need to know at least one point on each line.

One approach to finding a point on each line

For each of the two lines that contain the desired third vertices, N, of the triangle, there will be one vertex vertically above or below the point L. If you think of $\overline{LN}$ as the base of ΔLMN, then the height will be $v - c$.
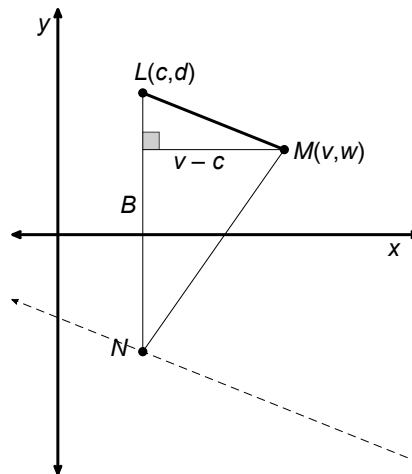
In order for ΔLMN to have an area of A, the length, B, of $\overline{LN}$ must satisfy
$$A = \tfrac{1}{2}B(v - c).$$
Therefore,
$$B = \frac{2A}{v-c}.$$

N will lie at this distance below L (as in the picture) or above it.

Consequently, on the line above L, the coordinates of N are

$$\left(c, d + \frac{2A}{v - c}\right),$$

and on the line below L, the coordinates of N are

$$\left(c, d - \frac{2A}{v - c}\right).$$

Using the points and the slope to find equations of the lines
If students understand point-slope form, they can use it to find the equations of the lines. An equation of the upper line is

$$y - \left(d + \frac{2A}{v - c}\right) = \frac{w - d}{v - c}(x - c).$$

This is a perfectly acceptable form of the equation. However, students with strong algebraic manipulation skills may be able to rewrite it in slope-intercept form as shown at the beginning of these Solutions:

$$y = \frac{w - d}{v - c}x + \frac{dv - cw + 2A}{v - c}.$$

If you apply the same process to the point on the lower line, the only change to either form of the equation is in the sign of $2A$.

Note: Students who are not familiar with point-slope form may try other methods such as finding the y-interept, b, by solving for it in this version of the slope formula

$$m = \frac{w - d}{v - c} = \frac{\left(d + \frac{2A}{v - c}\right) - b}{c - 0}.$$

However, these manipulations may be quite challenging!

What happens if L is at the origin
If L is at the origin, then $c = 0$ and $d = 0$. In this case, the equations simplify to

$$y = \frac{w - 0}{v - 0}x + \frac{0 \cdot v - 0 \cdot w \pm 2A}{v - 0} = \frac{w}{v}x \pm \frac{2A}{v},$$

which are the same as the equations in Problem #4 (with $w = y_1$ and $v = x_1$)!

Using the data from Problem #5

In Problem #5, the coordinates of T and U were (-6, 2) and (-2, 1), respectively, and the area was 9, meaning that

$$c = -6, d = 2, v = -2, w = 1, \text{ and } A = 9.$$

Substituting these into the equations discovered in this problem gives the results

$$y = \frac{w - d}{v - c}x + \frac{dv - cw \pm 2A}{v - c}$$

$$y = \frac{1 - 2}{-2 - -6}x + \frac{2 \cdot -2 - -6 \cdot 1 \pm 2 \cdot 9}{-2 - -6}$$

$$y = \frac{-1}{4}x + \frac{-4 - -6 \pm 18}{4}$$

$$y = -\frac{1}{4}x + \frac{2 \pm 18}{4}$$

Using the "+" sign:

$$y = -\frac{1}{4}x + \frac{2 + 18}{4} = -\frac{1}{4}x + 5$$

and using the "−" sign:

$$y = -\frac{1}{4}x + \frac{2 - 18}{4} = -\frac{1}{4}x - 4$$

The two equations agree with those found in Problem #5!

These comparisons to the results from Problems #4 and #5 provide strong evidence that the general equations discovered in this problem are likely to be correct.