

Deep Algebra Projects: Pre-algebra/Algebra 1

Stories That Formulas Tell

Topics

- Interpreting algebraic formulas
- Interpreting and calculating with negative numbers
- Evaluating algebraic expressions
- Order of operations
- Units and dimensional analysis

When scientists or mathematicians look at a formula, they see much more than a bunch of symbols or a recipe for calculating something. They see *stories about relationships*—stories that help them understand the connections between things.

Stories That Formulas Tell includes problems from everyday life, science, finance, sports, and geometry. There is a strong emphasis on science, because math is such a central part of science, and scientific formulas provide so many excellent examples of different mathematical relationships.

You may use this activity flexibly. Select any set of problems in order to focus on concepts or topics of your choice. For example, if your main goal is to deepen your students' understanding of negative numbers, you might assign Problems #1, #4, #7, and #8. The problems are challenging, so it is best to choose at least one or two of the earlier problems in order get students started successfully.

This activity has a slightly different format than others. There is a single page containing all of the formulas if you prefer to give them to students all at once. More importantly, there is just one page of Conversation Starters. It contains a single set of questions to use with every formula. I suggest leading a brief conversation before distributing this page. Begin by showing students the formulas from Problem #1. Ask them to “notice and wonder” as much as they can. Gently steer the conversation toward the questions on the Conversation Starters page. See how many of the questions your students can come up with on their own! Afterwards, give each student a copy of the Conversation Starters to use for the remainder of the project.

Students need not answer every question for every problem, but they should at least consider whether each question applies. Since every formula has its own unique

characteristics, different questions take center stage in each one. Of course, students should ask and answer their own questions as well!

The solutions in this project are longer than usual. There is so much to learn from studying even the simplest equations closely! Students will learn more from analyzing a few formulas in detail than from rushing to complete as many as possible.

In addition to the usual observations and strategies, the solutions include a lot of factual information that students are unlikely to know and cannot be expected to figure out for themselves. This is especially true at the beginning of each problem. After students have tried to predict the meaning of each variable and the purpose of the formula, you will need to give them this information in order for them to proceed.

After they have thought about a problem or a formula for a long time, you might like to share additional facts with them or encourage them to do some of their own research. The important thing is for them to do as much independent thinking as they can first.

Stories That Formulas Tell

Formula page

Stage 1

1. $d = rt$

$$F = ma$$

Daily life and science

2. $A = \pi r^2$

$$d = 16t^2$$

Geometry and science

3. $t = \frac{70}{I}$

$$A = 180 - \frac{720}{n}$$

Finance and geometry

Stage 2

4. $R = 0.3S - 0.6C$

Sports (baseball)

5. $S = 10 + \frac{D_1 + D_2}{2} - (X_1 + X_2)$

Sports (gymnastics)

6. $b = \frac{L}{4\pi d^2}$

$$T = \frac{2\pi}{\sqrt{g}}\sqrt{L}$$

Science (light and motion)

Stage 3

7. $v = v_0 + at$

Science (motion)

8. $t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$

Science (Einstein)

Conversation Starters for all formulas

*Ask questions like these in order to understand the story that a formula is telling.
Ask your own questions, too! Find your own new equations to explore!*

What do the variables stand for?

What do the subscripts (if any) mean?

What might the units (if any) of each variable be?

How do the units relate to each other?

Would the formula change if you changed the units? If so, how?

Why are certain quantities added? Subtracted? Multiplied? Divided?

If there is more than one formula, what do they have in common? How are they different?

Do certain symbols represent constant numbers that *never* change?

Do certain symbols represent numbers that remain constant in a particular situation?

When a variable's value changes, how does it affect the other variables' values?

Examples of general changes: increasing, decreasing.

Examples of specific changes: doubling, tripling, halving, adding 1.

Answer the same question if multiple values change simultaneously.

Can any of the variables be equal to 0? Which ones and why? What does this mean about the values of the other variables?

Can any of the variables be negative? Which ones and why? What does this tell you about the values of the other variables?

How do the answers to these questions relate to the real-world situation?

Are there any restrictions on the possible values of the variables? What causes these restrictions? Consider both real-world and mathematical causes.

What happens when certain variables take on extreme values? Consider both real-world and mathematical consequences.

Stage 1

In Problem #1, students investigate two simple formulas about motion. The first formula may be familiar, but the second formula may not. After students make predictions about what the formulas mean, share the *facts* about the variables and *units* with them so that they can move forward with further analysis. (See the beginning of the Solutions to find this information). Use the same approach for all problems in the activity.

Problem #2 introduces equations that involve a squared quantity. The first formula (for the area of a circle) may be familiar to them. Comparing and contrasting the two formulas will help students learn to see larger patterns in mathematical relationships.

The third problem asks students to deal with variables in denominators. For these formulas, increasing the value of one variable decreases another.

A key to success with these problems is *imagination*! If students feel stuck, they can feed their imagination by:

- (1) performing “thought experiments”
Change some aspect of the real-world situation, and imagine how it affects other aspects.
- (2) substituting lots of numbers
Change the numbers and observe how the other variables respond.

What students should know

- Be familiar with the standard order of operations.
- Know and use rules for computing with positive and negative numbers.
- Evaluate algebraic expressions.

What students will learn

- Use formulas to understand relationships between quantities.
- Use formulas to explore the effects of change.
- Interpret negative numbers in real-world contexts.
- Understand how units of measurement combine.
- Use math to explore new and challenging scientific and practical concepts.

Problem #1

daily life and science

$$d = rt$$

$$F = ma$$

Directions

- Try to predict what the formulas are about and what the variables mean.
- Answer questions like the ones on the Conversation Starters page.
- Ask and answer your own questions about the formulas.

Diving Deeper

Find other formulas that have a format similar to these. Compare and analyze them.

Solutions for #1

Facts

$d = rt$ motion in a straight line at constant speed

d : distance traveled

r : rate of travel

t : time traveled

$F = ma$ Newton's second law of motion

F : force

m : mass

a : acceleration

Sample choices of units

$d = rt$	<u>Customary units</u>	<u>Metric units</u>
rate:	feet/second	meters/second
time:	seconds	seconds
distance:	feet	meters

$F = ma$	<u>Metric units</u> (Customary units are rarely used by scientists.)
mass:	kilograms
acceleration:	meters/second ² (You can think of this as "(m per sec) per sec.")
force:	newtons (kilogram-meters/second ²)

Notice how the units multiply together. For example:

feet/second · seconds = feet

kilograms · meters/second² = kilogram-meters/second² (also called Newtons)

If you mix units (for example, if you use *meters* for d but *feet* per second for r , you need to put a number into the formula to make this conversion.

Common features of the formulas

Both formulas have the same format: one quantity is equal to the product of two other quantities. Because of this, the basic relationship between the quantities in one equation is the same as in the other.

Constants

Neither formula contains constants that can never change, but they do describe *situations* in which one of the variables usually stays constant. $d = rt$ typically deals with motion at a constant rate (r), while $F = ma$ describes the motion of an object with a constant mass (m). When r and m are constant, they are the rates of change (slopes) of linear relationships. (d is linear with regard to t , etc.)

There are unusual cases where m changes during motion. For example, a rocket's mass decreases during flight as it burns fuel. Or a snowball may gather more snow as it rolls down a hill. This complicates the process of calculating the motion. Also, Einstein discovered that an object's mass increases as its speed increases! The effect is so extremely small in everyday life that we never notice it. (See Problem #8.)

Effects of changing the values of variables

Note: Students should explore these kinds of changes by experimentation—by substituting lots of numbers. Then they should verify that their mathematical discoveries make sense in the real-world.

Assuming that the rate is constant, when you double t , d must double as well. This makes sense: if you travel twice as long at a constant speed, you will go twice as far.

Assuming that the mass is constant, when you double F , a must double as well. This makes sense, too. For a given mass, you must apply twice the force in order to produce twice the acceleration.

Another thought experiment:

Suppose you double the mass. If you keep the acceleration, a , the same, then F must double. In other words, it now takes twice as much force to produce the same acceleration. Since acceleration means change in motion, Newton's second law tells us that the greater the mass, the more an object *resists changing its motion*. (This resistance is called *inertia*.) Most of us think of mass as the amount of "stuff" an object has in it. A truer understanding is that mass is an object's resistance to change in motion.

Zero values

Any of the three quantities can be 0 in $d = rt$. If either r or t is 0, then d will be 0. If d is 0, then either r or t must be 0.

F or a can be 0 in $F = ma$. If either is 0, then the other must be 0 as well. It probably does not make sense for m to be 0. If it were, you could produce any acceleration with no force at all, because F would equal 0 no matter what value a had! (Physicists have discovered particles such as *neutrinos*, which at first appeared to be massless, but now have been discovered to have a very small mass. Particles of light, called *photons*, have no mass, but their motion is governed by other laws.)

Negative values

In $F = ma$, m must be positive. Negative mass makes no sense (except perhaps in exotic theories from modern physics). If you choose a certain direction as positive, then negative values of F represent a force pointing in the opposite of this direction (and similarly for acceleration). Since m is positive, F and a always have the same (+ or -) sign, meaning that they point the same direction. In other words, a force pointing in some direction produces an acceleration (though not necessarily a velocity!) in the same direction. If m were negative, applying a force in one direction would accelerate the object in the opposite direction!

In $d = rt$, any of the three quantities may be negative. Once you choose a positive direction, the opposite direction is negative. Suppose that you choose positive to the right, so that negative is to the left. If r is negative, the object is moving to the left. Multiplying r by a (positive) time gives a negative answer for d , meaning that the object is now located to left of where it started at $t = 0$.

You may choose any starting time to call $t = 0$. When $t = 0$, d must equal 0. In other words, $d = 0$ stands for your starting position. If r and t are both negative (and if the object has been moving the same way even before this time), then d is positive. This makes sense, because if the object is moving to the left, and you asking where it was *before* $t = 0$, It will have been to the *right* of the starting point!

Note: Students may also ask and answer their own questions! For instance, they may wonder how to calculate position if the rate is constantly changing.

Problem #2

geometry

$$A = \pi r^2$$

science

$$d = 16t^2$$

Directions

- Try to predict what the formulas are about and what the variables mean.
- Answer questions like the ones on the Conversation Starters page.
- Ask and answer your own questions about the formulas.

Diving Deeper

Find other formulas that have a format similar to these. Compare and analyze them.

Solutions for #2

Facts

$$A = \pi r^2 \quad \text{area of a circle}$$

A : area

π : a constant

r : radius

$$d = 16t^2 \quad \text{motion of a falling object}$$

d : distance fallen

16: a constant

t : time fallen

Sample units

$$A = \pi r^2$$

radius:

π

area:

Customary units

inches

none

square inches (in²)

Metric units

centimeters

none

square centimeters (cm²)

$$d = 16t^2$$

time:

16:

distance:

Customary units

seconds

feet per second² (ft/sec²)

feet

Notice how the units multiply together. For example:

$$\text{inches} \cdot \text{inches} = \text{inches}^2$$

$$\text{feet/second}^2 \cdot (\text{seconds})^2 = \text{feet}$$

The number 16 must have units of ft/sec² in order for the units to multiply properly as above. In fact, 16 ft/sec² is half of 32 ft/sec², which is the acceleration of an object falling to Earth (when air resistance is small). You may think of 32 ft/sec² as an increase in speed of 32 ft/sec every second.

The formula, $d = 16t^2$ is often written as $d = \frac{1}{2}gt^2$, where $g = 32 \text{ ft/sec}^2$ stands for the acceleration due to Earth's gravity. The factor of one-half shows up often in formulas that involve the square of a quantity.

If you use units other than feet (meters, for instance), the number 16 changes. This number also changes if you are on a different planet, because the force of gravity is different. The formula $d = \frac{1}{2}gt^2$ is more general than $d = 16t^2$, because it fits all of these situations.

Common features of the formulas

Both formulas have the same format: one quantity equals a constant times the square of another quantity. Because of this, the basic relationship between the quantities in one equation is the same as in the other.

Constants

The number, π , is a constant that never changes. It is a specific number whose value is always the same, regardless of the situation in which you use it.

The number, 16, is a constant that applies on the Earth when you use units of feet and seconds. As discussed above, this number will change if the units change (or the planet changes).

Effects of changing the values of variables

If you double the radius, the area multiplies by 4. If you triple the radius, the area multiplies by 9. In general, if you multiply the radius by n , the area multiplies by n^2 .

If you double the time, the object falls 4 times as far. In triple the time, it falls 9 times as far. In general, if you multiply the time the object has fallen by n , the distance it has fallen multiplies by n^2 .

These effects are the same for both formulas, because both formulas have the same form: a quantity equal to a constant multiplied by another quantity squared.

Zero value

A circle of radius 0 would not be very interesting; it would be a point with area 0!

In the second formula, $t = 0$ represents the time that an object begins its drop. If you choose to let $t = 0$ represent a different time, you will have to adjust the formula.

Negative values

A negative radius or area makes no sense in the case of $A = \pi r^2$.

A negative time in the second formula would represent a time before the object was dropped, which does not fit this real-world situation.

Note: Students may also ask and answer their own questions! For instance, they may want to explore how the number 16 changes on different planets or for different units. They may wonder how the whole formula changes if you include the effects of air resistance.

Problem #3

$$\begin{array}{c} \text{finance} \\ t = \frac{70}{I} \end{array}$$

$$\begin{array}{c} \text{geometry} \\ A = 180 - \frac{720}{n} \end{array}$$

Directions

- Try to predict what the formulas are about and what the variables mean.
- Answer questions like the ones on the Conversation Starters page.
- Ask and answer your own questions about the formulas.

Diving Deeper

Find other formulas that have a format similar to these. Compare and analyze them.

Solutions for #3

Facts

$$t = \frac{70}{I}$$

growth of money in a savings account

t : time required for the money to double

I : Interest rate

$$A = 180 - \frac{720}{n}$$

angle at the tip of a star formed from a regular polygon (See the pictures for a pentagon and hexagon.)

A : angle

n : number of sides in the polygon

Sample units

$$t = \frac{70}{I}$$

I :

70:

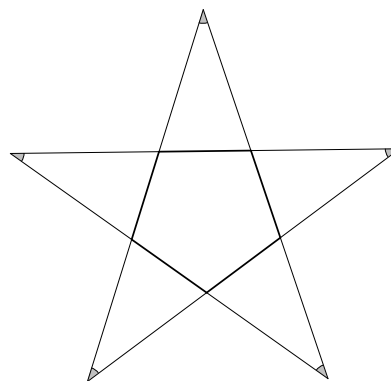
t :

Customary units

percent per year

percent

years



$$A = 180 - \frac{720}{n}$$

n :

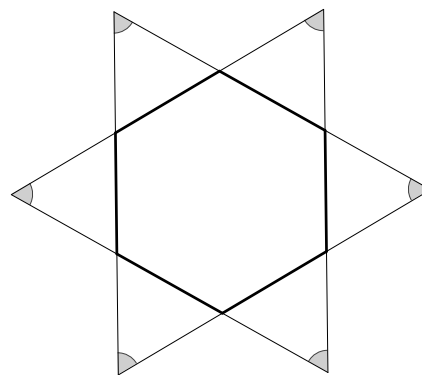
180 and 720:

A :

none

degrees

degrees



Notice how the units combine.

percent / (percent per year) = years

degrees – degrees = degrees

The second equation reinforces the idea that you always subtract like from like.

Note: Dividing a unit by a quantity that has no units leaves the unit unchanged.

The constant, 70, in the first formula stays the same as long as the two units of time are the same. However, if you were to use units of percent per *year* for I and wanted an answer in *months*, you would need to adjust the value of the numerator in order to make the conversion.

Common features of the formulas

Both formulas contain a fraction with a constant in the numerator and a variable in the denominator. This format has a big impact on how the values of the quantities change. The second formula is different in that the fraction is subtracted from a constant.

Constants

The first formula is approximate, because the number 70 is approximate. Students will learn in high school that this number comes from the *natural logarithm* of 2 ($\ln 2$), which is approximately 0.693. It is expressed as a percentage, because the interest rate is also expressed as a percentage. You could just as well express both numbers in the fraction as decimals. The whole number (percentage) version is probably easier.

Students will learn another unit (*radians*) for measuring angles when they study trigonometry. Using radians would require 180° and 720° to be expressed differently (as π and 4π).

Effects of changing the values of variables

In the first formula, as I increases, t decreases and vice versa. This makes sense, because you would expect the doubling time to decrease when you have a higher interest rate. More specifically, when interest rate, I , doubles, the time, t , becomes half as long. In general, when you multiply I by n , the time, t , multiplies by $1/n$.

In the second formula, as n increases, $720/n$ decreases. Since you are subtracting a decreasing fraction from 180° , the answer is getting closer to 180° . This makes sense. Imagine extending the sides of a regular polygon. As you increase the number of sides of the polygon, the angles in the star-tips “flatten out” and approach a straight angle. (The sides of the stars also get closer and closer to the sides of the polygon.)

Zero values

If the interest rate were 0%, the money would never double. This is reflected in the fact that division by 0 is undefined. As the interest rate gets smaller and smaller (approaching 0), the doubling time increases without limit. Some students may imagine an infinite doubling time.

On the other hand, there is no interest rate high enough to make the doubling time equal to 0. As the interest rate increases without limit, the doubling time gets closer and closer to 0 but never gets there.

Negative values

Negative interest rates and doubling times do not make sense. Polygons with a negative number of sides do not make sense either. However, the second formula will give a zero or negative value for A for some polygons. See the discussion below.

Constraints on the values of the variables

In the first formula, I and t may take on any positive values. Extreme values are probably not realistic, but it is hard to define a clear cutoff point between reasonable and unreasonable values.

In the second formula, n must be a whole number greater than or equal to 3. (There are no polygons with fewer than 3 or a fractional number of sides.) Of course, it is possible to substitute other values into the equation, but the answers will not be meaningful in this situation.

Interestingly, if you substitute $n = 3$ or $n = 4$ into the equation, you get answers of -60° and 0° respectively for A , which do not appear to be reasonable angles for the tips of a star. This suggests that you cannot make a star by extending the sides of an equilateral triangle or a square, which is true! However, some students may be able to come up with creative meanings for the values -60° and 0° by visualizing what happens to the extended sides as n gets smaller and smaller.

There is no limit on how large n can be. As it becomes extremely large, A will be virtually indistinguishable from 180° , and the star will look more and more like the polygon it is made from, which in turn will look more and more like a circle!

Note: Students may also ask and answer their own questions! For example, they may wonder if it is possible to find formulas for tripling or quadrupling times.

Stage 2

The formulas in Stage 2 look quite a bit more complex than those in Stage 1, but this is only because they contain more “pieces.” In particular, most of them have more variables. The concepts underlying the formulas are not much more difficult than those in Stage 1. The key is for students to identify the pieces, examine them separately, and then put them together.

What students should know

- Be familiar with the standard order of operations.
- Know and use rules for computing with positive and negative numbers.
- Evaluate algebraic expressions.

What students will learn

- Use formulas to understand relationships between quantities.
- Use formulas to explore the effects of change.
- Interpret negative numbers in real-world contexts.
- Understand how units of measurement combine.
- Use math to explore new and challenging scientific and practical concepts.

These prerequisites and goals are the same as in Stage 1, but the formulas are more complex.

Problem #4

sports (baseball)

$$R = 0.3S - 0.6C$$

Directions

- Try to predict what the formula is about and what the variables mean.
- Answer questions like the ones on the Conversation Starters page.
- Ask and answer your own questions about the formula.

Solutions for #4

Facts

$R = 0.3S - 0.6C$ the “stolen base runs” statistic in baseball

R : stolen base runs statistic

S : number of stolen bases

C : number of times caught stealing base

The *stolen base runs* statistic, R , predicts how attempts to steal base affect the overall number of runs that a team scores.

Units

The variables and numbers have no units. S and C just count the number of times a base is stolen or the runner is caught. R is just a number that combines S and C in a meaningful way using the numbers 0.3 and 0.6.

The meanings of the operations

Students must know the standard order of operations in order to interpret this formula correctly.

The right side of the formula comes in two “pieces” called *terms*. The first term, $0.3S$, describes the effect of successfully stolen bases. The second, term, $0.6C$, describes the effect of a runner getting caught stealing. The fact that the second term is subtracted shows that getting caught decreases the value of R . On the other hand, when S increases, R increases with it. Since stealing base successfully is better than getting caught, it appears that high R scores are more desirable than low R scores (as you might expect).

Constants

The constants 0.3 and 0.6 have no obvious meaning by themselves, but multiplying each variable by its constant gives the appropriate *weights* to S and C . Since 0.6 is greater than 0.3, the variable C has greater power than does S to affect the value of R . In other words, getting caught stealing hurts more than a successful steal helps.

Effects of changing the values of variables

When you increase S by 1, the value of R increases by 0.3. When you increase C by 1, the value of R decreases by 0.6. This shows the relative effects of S and C very precisely.

Zero values

S and C can equal 0 (both or separately).

R will equal 0 whenever the number of successful steals is twice the number of times that the runner gets caught. It may take a little time and effort for students to reach this conclusion. They may do it (1) intuitively, (2) by substituting numbers, or (3) by solving the equation $0.3S - 0.6C = 0$ for S or C .

Example:

If $S = 8$ and $C = 4$ then:

$$\begin{aligned} R &= 0.3 \cdot 8 - 0.6 \cdot 4 \\ &= 2.4 - 2.4 \\ &= 0 \end{aligned}$$

In general, if $S = 2n$ and $C = n$, then:

$$\begin{aligned} R &= 0.3 \cdot (2n) - 0.6n \\ &= (0.3 \cdot 2)n - 0.6 \cdot n \\ &= 0.6n - 0.6n \\ &= 0 \end{aligned}$$

It seems reasonable to guess (and it is true!) that an R value of 0 means that attempted steals have had a neutral effect on a team's number of runs. Thus, $S = 2C$ represents a sort of "break-even" point.

Negative values

It does not make sense for S or C to be negative, but R may be either positive or negative.

If $S > 2C$, then R is positive, meaning that the team's (or a player's) net number of runs has probably increased due to stealing.

If $S < 2C$, then R is negative, meaning that the team's (or a player's) net number of runs has probably decreased due to stealing.

Constraints on the values of the variables

S and C must both be whole numbers. R will always be a multiple (positive or negative) of 0.3. Large absolute values of S , C , and R are unlikely, because of the limited number of times that a player or a team is likely to attempt stealing base. However, there is no clearly defined limit to these values.

General observations

Base-stealing has become less frequent in professional baseball in recent years, probably because the mathematics has shown that it may be less beneficial than previously believed. Teams are likely to be more strategic by focusing their stealing attempts on times when the benefits are likely to be greater (perhaps when stealing 3rd base, when bases have more runners, etc.). They may also restrict it to players who are successful often enough for it to be helpful.

Note: Students may also ask and answer their own questions! For example, they may wonder how the formula was created. Or they may wonder what percentage of base stealing attempts must be successful in order to benefit the team. Encourage them to figure this out! (The answer is about 67% [two-thirds of the time]. This is based on the break-even number. A typical rule of thumb used in baseball is 70%.)

Problem #5

$$S = 10 + \frac{\text{sports (gymnastics)} \quad D_1 + D_2}{2} - (X_1 + X_2)$$

Directions

- Try to predict what the formula is about and what the variables mean.
- Answer questions like the ones on the Conversation Starters page.
- Ask and answer your own questions about the formula.

Solutions for #5

$$S = 10 + \frac{D_1 + D_2}{2} - (X_1 + X_2)$$

gymnastics vault score (2013 – 2016)

Facts

S : overall score for two vaults

10 and 2: constants

D_1 : difficulty score for first vault

D_2 : difficulty score for second vault

X_1 : execution deductions for first vault

X_2 : execution deductions for second vault

Units

This formula has no units (except perhaps *points*). Each number represents a score of some type.

The meanings of the operations

The formula looks complicated, because it is so long. However, if you look at it in three separate pieces, it is not so confusing!

First, 10 is added at the beginning, because every vaulter begins with a score of 10. In fact, according to the formula, if you did not perform a vault at all, you would receive a score of 10, because D_1 , D_2 , X_1 , and X_2 would all equal 0! (I suspect that gymnastics officials would probably not apply the formula in this case...)

The fraction in the middle handles the points awarded for difficulty. The more difficult a vault is, the more points you are awarded. This is sometimes called a “starting value” for a vault. Adding the two starting values, D_1 and D_2 , and dividing by 2 simply gives the average of the two starting values. This average is added to the 10 points that every vaulter receives.

The $X_1 + X_2$ in parentheses at the end of the formula represents the total deductions for errors in the gymnast’s execution of the vaults. It is subtracted in the formula because deductions decrease a gymnast’s score.

In summary, the formula gives each gymnast 10 points to start with, adds the average of the difficulty points (“starting values”) for each vault, and subtracts all of the deduction points.

Constants

The constant, 10, could be changed if the gymnastics scoring committee decided to start gymnasts with a different base score. (If they removed the constant altogether, gymnasts would receive a negative score whenever their deductions exceeded the average of their difficulty points!)

The constant, 2, would probably not be changed, because it is used in the averaging process.

Effects of changing the values of the variables

The effect of changing the value of X_1 or X_2 is very direct. The gymnast's score decreases by the same amount that X_1 or X_2 increases.

The effect of changing the value of D_1 or D_2 is a little less straightforward. Since

$$\frac{D_1 + D_2}{2} = \frac{D_1}{2} + \frac{D_2}{2},$$

increasing D_1 or D_2 by some amount increases the gymnast's score by only half that amount. In other words, the effect of averaging the two values of D is to halve the amount that each D value adds to the total score.

Constraints on the values of the variables

D_1 , D_2 , X_1 , and X_2 are all positive numbers. The possible values for these variables are determined by the people who design the scoring process. They determine the starting (difficulty) values for each type of vault and the amount that is deducted for each error in execution.

Other observations and questions

Before 2013, each vault was scored separately ($10 + D - X$), and then the scores for the two vaults were averaged. What would this formula look like? Which formula results in higher scores? How much higher? Why? What has been the overall effect of the change? Why do you think the change might have been made?

Note: Students may also ask and answer their own questions!

Problem #6

science (light)

$$b = \frac{L}{4\pi d^2}$$

science (motion)

$$T = \frac{2\pi}{\sqrt{g}}\sqrt{L}$$

Directions

- Try to predict what the formulas are about and what the variables mean.
- Answer questions like the ones on the Conversation Starters page.
- Ask and answer your own questions about the formulas.

Solutions for #6

Facts

$$b = \frac{L}{4\pi d^2} \quad \text{brightness of a light source}$$

b : apparent brightness

L : luminosity

4π : constant

d : distance from source

$$T = \frac{2\pi}{\sqrt{g}}\sqrt{L} \quad \text{pendulum motion}$$

T : *period* of the pendulum (time for one cycle)

2π : constant

g : gravitational acceleration constant

L : length of the pendulum

Sample units

$$b = \frac{L}{4\pi d^2}$$

Metric units

Apparent brightness: watts per square meter

Luminosity: watts

4π : no units

Distance from source: meters

$$T = \frac{2\pi}{\sqrt{g}}\sqrt{L}$$

Period: seconds

2π : no units

Gravitational acceleration: meters per second²

Length of the pendulum: meters

Notice how the units combine.

$$\text{watts} \div ((\text{no units}) \cdot \text{meters}^2) = \text{watts} \div \text{meters}^2 = \text{watts per square meter}$$

The second one is complicated!

$$\frac{(\text{no units})}{\sqrt{\frac{\text{meters}}{\text{seconds}^2}}} \cdot \sqrt{\text{meters}} = \sqrt{\text{meters}} \div \frac{\sqrt{\text{meters}}}{\sqrt{\text{seconds}^2}} =$$

$$\frac{\sqrt{\text{meters}}}{1} \cdot \frac{\sqrt{\text{seconds}^2}}{\sqrt{\text{meters}}} = \sqrt{\text{seconds}^2} = \text{seconds}$$

Comparing the formulas

These formulas look more complex than most of the others up to this point. They both involve multiplying a constant by a variable expression. The variable expressions have different forms. In the first formula, there is a variable, L , in the numerator, and a second variable, d , which is squared and in the denominator. In the second formula, there is a single variable underneath a square root.

Constants

Each constant has multiple parts. In the first formula, the constant is $\frac{1}{4\pi}$. In the second formula, it is $\frac{2\pi}{\sqrt{g}}$. If you write each constant as a single (approximate) number, the formulas look simpler—especially the second one!

$$\text{first formula: } b \approx 0.0796 \frac{L}{d^2}$$

$$\text{second formula: } T \approx 2.006\sqrt{L}$$

Both formulas are approximate, because the decimals have been rounded. The second formula uses the metric value of $g \approx 9.807 \text{ m/sec}^2$ (on the Earth). (Remember that L means something different in each formula!)

The presence of π in these formulas may make students wonder if they have something to do with circles or spheres. The answer is yes. In the first formula, the light from a source is spreading out with the same intensity in all directions and equal at a given distance from the source—and the collection of points that is equidistant from the source is the surface of a sphere. In fact, the expression $4\pi d^2$ in the denominator is the surface area of the sphere at a distance of d from the light source! As this surface become larger and larger, the light spreads out over a greater area and thus becomes dimmer.

The connection to π is less obvious for the second formula. You might think that it is related to the fact that a pendulum's mass travels in a circular arc, but the real reason is that the mass is traveling (approximately) in *harmonic motion*. Harmonic motion often occurs in a straight line rather than a circular arc! The position an object in harmonic motion is described by a single coordinate of an imaginary object moving on a circular path at a constant rate.

Effects of changing the values of the variables

Examples for the first formula:

- If you double the luminosity (L), the apparent brightness doubles as well.
- If you double the distance (d) from the light source, the apparent brightness becomes one-fourth of what it was!
- If you triple the distance from the source, you will have to make the light source 9 times brighter in order to keep the apparent brightness the same!

Students should experiment with many ways of adjusting the two variables both separately and together.

Examples for the second formula:

- If you double the length of the pendulum, the period becomes $\sqrt{2}$ (about 1.4) times longer.
- In order to double the period of the pendulum, you must quadruple its length.
- In order to triple the period of the pendulum, you must make it 9 times longer!
- In general, in order to make the period n times longer, you must make the pendulum n^2 times longer.

Zero and extreme values

If the luminosity, L , in the first formula is 0, then the apparent brightness is 0 as well (which is not a very interesting situation). It does not make sense for the distance, d , to equal 0. As the distance from the source becomes smaller and smaller, the formula shows that the apparent brightness increases without limit! This seems strange and suggests that the formula may not be accurate for very small values of d . On the other hand, if d becomes extremely large, the apparent brightness dims, approaching 0 (if the luminosity does not change), which certainly makes sense.

Formulas like this in which a quantity is equal to something divided by the square of some variable are known as *inverse square laws*. Other examples of inverse square laws include the gravitational force generated by a mass and the electromagnetic force generated by a charge.

In the second formula, the length and period could both be very close to 0, but if they were equal to 0, there would be no pendulum! The formula does show, however, that as the pendulum gets shorter, its period decreases, meaning that it swings faster. It is probably reasonable to expect that the formula is less accurate in extreme situations, though. By the way, notice that the formula does not contain a variable for the mass. Apparently, the mass does not affect the period! You might try making pendulums to test this!

Stage 3

The formula for Problem #7 looks simpler than many of the formulas from earlier problems, but analyzing the various possibilities for the positive and negative values of each variable takes a lot of time and careful thought. Students should spend plenty of time on this and explain their thinking as carefully as they can.

Problem #8 contains Einstein's famous *time-dilation* equation that shows how time appears to slow down when you move faster. The formula requires a deep and careful analysis, but the payoff is worth it! Students get a glimpse of the mathematics behind one of the important, influential, and mind-bending scientific ideas of the 20th century.

What students should know

- Be familiar with the standard order of operations.
- Know and use rules for computing with positive and negative numbers.
- Evaluate algebraic expressions.

What students will learn

- Use formulas to understand relationships between quantities.
- Use formulas to explore the effects of change.
- Interpret negative numbers in real-world contexts.
- Understand how units of measurement combine.
- Use math to explore new and challenging scientific and practical concepts.

These prerequisites and goals are the same as in Stages 1 and 2, but the analysis required is more deep and complex.

Problem #7

science (motion)

$$v = v_0 + at$$

Directions

- Try to predict what the formula is about and what the variables mean.
- Answer questions like the ones on the Conversation Starters page.
- Ask and answer your own questions about the formula.

Testing the Waters

Analyze the formula $x = x_0 + vt$. Note: This formula has the same format as above, but v is replaced by x (the *position* of the object), and a is replaced by v . The mathematical ideas are the same, but the situation is easier to visualize.

Diving Deeper

- Imagine the details of a real-world situation in which the formula would apply. What is the object? What is causing the constant acceleration?
- Analyze the straight-line motion formula, $x = v_0t + \frac{1}{2}at^2$, where x stands for the object's position. Focus on the meanings of negative values for each variable.

Solutions for #7

Facts

$$v = v_0 + at$$
 velocity of motion with constant acceleration

v : velocity (at the time t)

v_0 : initial velocity (when $t = 0$)

a : acceleration

t : time

Sample units

$$v = v_0 + at$$

velocity:

Customary units

feet per second

Metric units

meters per second

initial velocity:

feet per second

meters per second

acceleration:

feet per second²

meters per second²

time:

seconds

seconds

Notice how the units combine.

$$\text{ft/sec} + \text{ft/sec}^2 \cdot \text{sec} = \text{ft/sec} + \text{ft/sec} = \text{ft/sec}$$

Values that you add (or subtract) in a formula must always have like units.

Constants

a and v_0 may be any numbers, but in a given real-world situation, they are constant: there can be only one starting velocity, and once you have a value for the acceleration, the object always accelerates at that rate. (This assumption is built into the formula. If the acceleration changed while the object moved, it would not make sense to simply multiply the acceleration by the time.)

You do not need (constant) numbers in the formula unless the units don't match. For example, if you used ft/sec^2 for the acceleration (a) but used *minutes* for the time (t), you would need to multiply the acceleration \cdot time (at) term by 60 in order to convert t into seconds. The formula would become $v = v_0 + 60at$. (This would be a very unconventional thing to do!)

Effects of changing the values of the variables

As stated above, the acceleration (a) and the initial velocity (v_0) are constant in any given problem situation. This leaves t and v as the variables. If you think naturally of t as the input and v as the output, you have a linear relationship with a starting value (y -intercept) of v_0 and a rate of change (slope) of a . (The language that you and the students use to talk about this will depend on their previous experience.)

The input is the variable that you naturally think of changing. Whenever t increases by 1, v changes by a . (This is the definition of the rate of change!)

Zero values

Any of the quantities may equal 0.

If $v_0 = 0$, it simply means that the object starts at rest. In that case, the relationship is proportional.

If $a = 0$, then the object is not accelerating. It is moving at the constant speed, v_0 .

When $t = 0$, the value of v is equal to v_0 . You can see this at least two ways: (1) This is the *definition* of initial velocity, and (2) if you substitute 0 for t in the formula, you are left with $v = v_0$.

It is possible for v to equal 0 one time during the object's motion. Whenever the acceleration acts in a direction opposite to the object's velocity, the object will slow down. If this continues for long enough, it will eventually stop. (If it continues even longer, the object will reverse direction and begin speeding up in that direction.)

Negative values

Any of the values can be negative, because you can choose the time that $t = 0$ represents and the direction represented by positive velocity and acceleration (in which case negative numbers stand for the opposite direction). In the examples below, we will assume that positive values of v_0 and a are to the right and negative values are to the left.

Examples of negative values: (This analysis requires patience and careful thought!)

(1) v_0 and a are both positive.

The object is moving to right and accelerating to the right. In other words, at $t = 0$, the object begins with a rightward speed v_0 and moves faster and faster.

If t is negative, you are looking at times earlier than $t = 0$ when the object will have been moving more slowly than v_0 . This makes sense, because acceleration \cdot time will be negative ($+ \cdot - = -$), so you will be adding a negative value to v_0 .

(2) v_0 is negative and a is positive.

When v_0 and a have opposite signs, the object is slowing down. In this case, the object is moving left and accelerating to the right. In other words, at $t = 0$, the object begins with a leftward speed v_0 but is slowing down. If the motion continues long enough, the object will eventually stop and begin moving to the right. The change in direction occurs when $|a \cdot t| > |v_0|$.

If t is negative, you are looking at times earlier than $t = 0$ when the object will have been moving faster than v_0 . This makes sense, because acceleration \cdot time will then be negative ($+ \cdot - = -$), so you will be adding a negative value to the negative value of v_0 , which will make it even more negative (representing faster motion in the negative direction).

(3) v_0 is positive and a is negative.

The object is initially slowing down, because v_0 and a have opposite signs. In this case, the object is moving right and accelerating to the left. In other words, at $t = 0$, the object begins with a rightward speed v_0 but is slowing down. If the motion continues long enough, the object will eventually stop and begin moving to the left. The change in direction occurs when $|a \cdot t| > |v_0|$.

If t is negative, you are looking at times earlier than $t = 0$ when the object will have been moving faster than v_0 . This makes sense, because acceleration \cdot time will then be positive ($- \cdot - = +$), so you will be adding a positive value to the positive value of v_0 , which will make it even more positive, representing faster rightward motion.

(4) v_0 and a are both negative.

The object is moving left and accelerating to the left. In other words, at $t = 0$, the object begins with a leftward speed v_0 and moves faster and faster.

If t is negative, you are looking at times earlier than $t = 0$ in which the object will have been moving more slowly than v_0 . This makes sense, because acceleration \cdot time will then be positive ($- \cdot - = +$), so you will be adding a positive value to the negative value of v_0 , making it less negative (slower moving in the leftward direction).

Problem #8

science (Einstein)

$$t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

Directions

- Try to predict what the formula is about and what the variables mean.
- Answer questions like the ones on the Conversation Starters page.
- Ask and answer your own questions about the formula.

Solutions for #8

$$t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

Einstein's time dilation formula

Facts

t : a time interval

t' : the dilated time interval

1: constant

v : velocity

c : speed of light

This is one of Einstein's special relativity formulas. It describes the way that time "slows down" for moving objects. t stands for a time interval as measured by an observer moving with velocity, v . t' represents the same interval as seen by a *stationary* observer (for whom $v = 0$).

Sample units

time intervals (t' and t) years

Any unit of time works as long as it is the same for both t' and t .

1 (constant) no units

velocity (v) meters per second

speed of light (c^*) meters per second

Again, any unit of velocity works as long as it is same for both v and c .

Notice how the units combine:

When you divide v by c , you get an expression with no units. This makes sense, because you are subtracting it from the constant, 1, which has no units.

(Remember that the units of quantities that you subtract must be the same.)

Therefore, the entire denominator has no units. And when you divide the time (t) by an expression with no units, the result still has units of time.

Although $\frac{v}{c}$ has no units, it does have an important meaning. It is a ratio that represents the *fraction of the speed of light* that you are traveling. For instance, if you are traveling at 75% of the speed of light, then $\frac{v}{c} = 0.75$. As a result, you often do not need to know the values of v and c separately. You may simply express your speed using this ratio.

*Some students may recognize the c from Einstein's formula, $E = mc^2$.

Constants

The number 1 underneath the square root is constant in all situations.

The speed of light (c) is approximately equal to 3×10^8 m/sec. Two of Einstein's many great achievements were (1) to recognize that c is in fact constant for all observers traveling at any speed, and (2) to explore the implications of this fact—one of which is the strange time dilation formula in this problem!

Effects of changing the values of the variables

As you change the velocity, v (or equivalently, the ratio $\frac{v}{c}$) it alters the relationship between t and t' in a fairly complex way—as you can see from the complicated expression in the denominator!

You can explore the relationship by trying many values of v (or $\frac{v}{c}$) and watching what happens to t and t' . To simplify the process, try setting $t = 1$. Then t' is the factor by which t increases. For example, if $t' = 2$, the time interval doubles.

Zero values

If $t = 0$, then $t' = 0$ as well, but this is not very interesting.

If $v = 0$, then the denominator is equal to 1 (try it!), and $t = t'$. In other words, time-interval measurements are the same for both observers.

Negative values

v could be negative, but this is not important, because the direction of travel does not matter. Since $\frac{v}{c}$ gets squared in the calculations, the result will always be positive anyway!

t and t' must always have the same sign, + or −*, because the denominator (being an expression beneath a $\sqrt{\quad}$ symbol) is always positive. Encourage students to think about why this is a good thing! First, make sure they understand that t and t' represent time *intervals*—a *difference* in time measurements between two events.

If t and t' had opposite signs, it would mean that the individual times measured by each observer for the two events occurred in reverse order. In other words, the observers would disagree about which event happened first! This would cause serious problems with the idea of cause and effect.

* It is natural to think of them as positive numbers.

Extreme values

In all of our everyday life experiences, v is *extremely* small (that is—compared to the speed of light). In this case, $\frac{v}{c}$ is *very* close to 0, meaning that the denominator in the formula is *very* close to 1. Therefore, t' is so close to t that it is nearly impossible for us to measure the difference. In other words, the speeds that we encounter in our daily lives are nowhere near large enough for us to see time appear to slow down.

Example: Suppose that $v = 1000$ miles per hour. This is about 450 meters per second. Remembering that c is approximately 3×10^8 meters per second:

$$\frac{v}{c} = 450 \div (3 \times 10^8) \approx 1.5 \times 10^{-6} = 0.0000015$$

$$\left(\frac{v}{c}\right)^2 \approx 2.25 \times 10^{-12} = 0.00000000000225$$

$$1 - \left(\frac{v}{c}\right)^2 \approx 0.999999999998$$

The square root of this number is even closer to 1! In fact, your calculator will probably not have enough precision to distinguish it from 1.

In order to calculate t' , you must divide t by this number, which, for all practical purposes, amounts to dividing by 1. The amount by which time appears to slow down is difficult to calculate and too small to measure.

Suppose that you travel at the speed of light. In other words, $v = c$. Then:

$$\frac{v}{c} = 1$$

$$\left(\frac{v}{c}\right)^2 = 1$$

$$1 - \left(\frac{v}{c}\right)^2 = 1 - 1 = 0$$

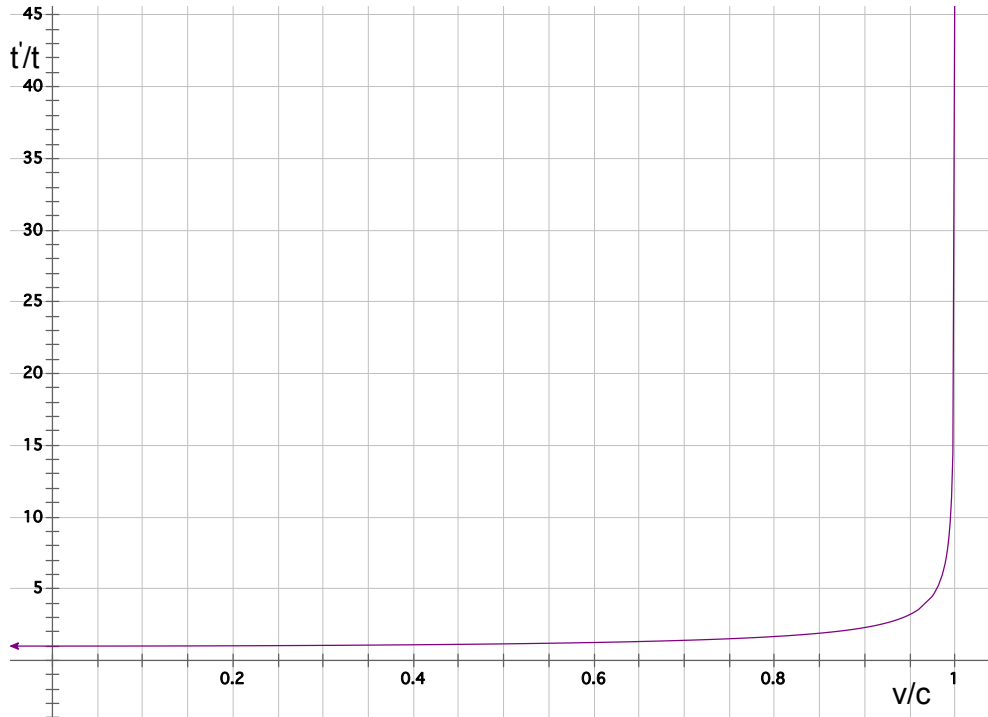
$$\sqrt{0} = 0$$

The denominator in the formula is equal to 0, which makes no mathematical sense. The formula appears to be telling us that it is not possible to travel at the speed of light! (The problem is as bad or worse if $v > c$. Try it!)

What happens when v becomes very close to c ? Then $\frac{v}{c}$ becomes very close to 1, and the denominator gets extremely small. When you divide t by this number, you get a very large answer, meaning that t' is much greater than t ; that is, the time dilation effect is huge. When you are traveling near the speed of light, tiny increases in v eventually result in enormous increases in t' . This helps to explain the problem with having a denominator of 0. If v were to reach the value of c , the value of t' would essentially be infinite.

A graph

Some students may wonder what the relationship between v and t' (or more specifically $\frac{v}{c}$ and $\frac{t'}{t}$) looks like on a graph. Encourage them to try graphing it!



Notice that up to half the speed of light ($\frac{v}{c} = 0.5$) and even beyond, the curve is very flat, showing that the time dilation effect is small. (Everyday life is right next to the y -axis!) When you reach about 0.95 times the speed of light, the curve bends very sharply upwards and the time dilation effect suddenly becomes dramatic. The graph never touches the vertical $\frac{v}{c} = 1$ line, but gets closer and closer.

Additional observations

In truth, you cannot determine who is stationary and who is moving in Einstein's theory. You can only determine the *relative* motion of the two observers. Since either observer may claim to be at rest, each of them may conclude that time appears to be moving more slowly for the other! This paradox takes a lot of thought to work out. Some students may be interested in doing some research.

The expression $\sqrt{1 - \left(\frac{v}{c}\right)^2}$ appears in many of Einstein's formulas, including those related to length and mass at high velocities. It is not a coincidence that the denominator looks suspiciously like the familiar expression, $\sqrt{a^2 + b^2}$, used in Pythagorean theorem calculations. The main difference is between the addition and subtraction in the formulas, which reflects the fact that time behaves differently than the familiar spatial dimensions.