Deep Algebra Projects: Pre-Algebra/Algebra
Multiplication Table Algebra

Topics

- Distributive, Associative, and Commutative Properties of Multiplication
- Using variables to represent and analyze patterns
- Equivalent algebraic expressions
- Simplifying algebraic expressions by combining like terms
- Multiplying polynomials

You may think of multiplication tables as “grade-school arithmetic,” but they contain a treasure trove of complex and subtle patterns that students can explore algebraically. The *Multiplication Table Algebra* project introduces a variety of the such patterns and invites students to discover and analyze some of their own!

Stage 1

Note: A blank copy of a multiplication table is available on Handout #1 near the end of this project. Students may use it to further explore questions in Problems #1 and #2 or to discover and prove some of their own patterns.

What students should know

- Describe patterns using algebraic patterns.
- Simplify polynomials by combining like terms
- Multiply simple polynomials, including binomials by binomials.

What students will learn

- Recognize, extend, and describe complex patterns.
- Use algebraic processes to prove conjectures.
- Deepen knowledge of known algebraic relationships and discover new ones.

Problem #1

·	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

·	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Directions

- Describe patterns involving the sets of three shaded squares.
- Generalize the patterns.
- Choose a "base square" to represent row a and column b, and prove the generalized pattern algebraically.
- Choose a different base square and repeat the proof.

Solutions for #1

Notes

In the Solutions for this problem, a represents the number in the left column of the multiplication table, and b represents the number in the top row.

Students may have many insightful observations that are not in these Solutions.

Describing the patterns

Left picture:

- The sum of each pair of numbers in the '2' column equals the corresponding number in the '4' column.
- The number between each pair of numbers in the '2' column equals the average of the two numbers.
- The number in the '4' column is double the number in between the pair in the '2' column.
- All three "triangles have the same shape and size.
- If you move the triangles up or down any number of squares, the patterns continue to hold true.
- If you rotate the triangles clockwise 90°, the patterns continue to hold true.

Right picture:

- The patterns from the left picture still apply, except that you can replace all statements about the '2' column by the '3' column and all statements about the '4' column by '8' column.

Generalizing the patterns

You may create triangles like these beginning in any column *n*. The right "vertex" of the triangle will always be (1) in the row between the two rows in column *n*, and (2) in column 2*n*.

In other words, as you move the triangles to the right, they become more "stretched out." Their rightmost column is *n* squares to the right of the two squares in the left column.

Proving the patterns algebraically

For each triangle, suppose that the square between the two left squares is said to be the "base square" (sitting in row a and column b). Then the number in that square is ab, and the numbers in the triangle are represented by the algebraic expressions:

Top left square (TL)	$(a-1)b$
Bottom left square (BL)	$(a+1)b$
Right square (R)	$a(2b)$ or simply $2ab$

The goal is to prove is that TL + BL = R.

TL + BL =
$(a-1)b+(a+1)b=$
$ab-b+ab+b=$
$2ab=$
R

You may also use these calculations to verify that the number between the TL and BL squares is their mean:

Mean of TL and BL =
(TL + BL) / 2 =
$\frac{2ab}{2}=$
ab

Choosing a different "base square"

Suppose you choose TL as the base square. Then TL sits inn row a and column b. In this case:

TL + BL =	R =
$ab+(a+2)b=$	$(a+1)(2b)=$
$ab+ab+2b=$	$a(2b)+1(2b)=$
$2ab+2b=$	$2ab+2b$

Since TL and R are equal to the same expression, they must be equal to each other.

Students may make other choices for the base square. However, they must be careful that their algebraic expressions capture *general* relationships that apply to all of the triangles.

Problem #2

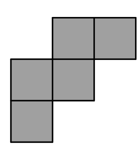
·	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Directions

- Describe one or more patterns involving sums of numbers in each set of shaded squares. Prove the pattern(s) algebraically.
- Describe one or more patterns involving products of numbers in each set of shaded squares. Prove the pattern(s) algebraically.
- On a blank multiplication table, shade four squares at the corner of a rectangle. Repeat many times. Find and prove one or more patterns.

Diving Deeper

- Place copies of the shape



  in different places on the multiplication table. Look for patterns and prove them algebraically.
- Create your own shapes to place over and move around the multiplication table. Look for patterns and prove them algebraically.

Solutions for #2

Throughout the Solutions for #2, the numbers in the shaded squares are abbreviated.

Upper left: UL Upper right: UR

Center: C

Lower left: LL Lower right: LR

Patterns involving sums

UL + LR is always two greater than twice C. UL + LR = 2C + 2

UR + LL is always two less than twice C. UR + LL = 2C – 2

The sum of the four "diagonal" numbers is always four times the center number.

Some students may note that the pairs directly above/below and left/right of the center number have sums exactly *equal* to twice the center number! (They may find many other patterns as well.)

Proofs of the patterns

It seems natural to select the center square as the "base square," in which case C = ab.

UL + LR =
$(a-1)(b-1)+(a+1)(b+1)=$
$(ab-a-b+1)+(ab+a+b+1)=$
$2ab+2=$
2C + 2

UR + LL =
$(a-1)(b+1)+(a+1)(b-1)=$
$(ab+a-b-1)+(ab-a+b-1)=$
$2ab-2=$
2C – 2

UL + LR + UR + LL = (2C + 2) + (2C – 2) = 4C

Patterns involving products

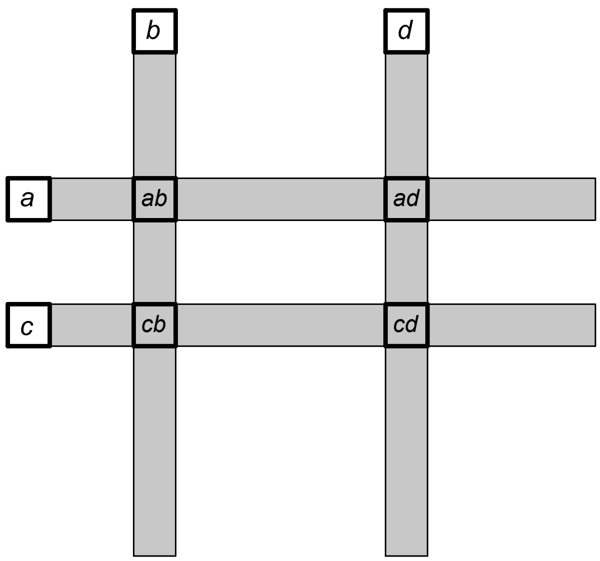
Opposite corners have the same product. In other words:

UL • LR = UR • LL.

This pattern is not limited to the types of pictures shown. It applies to any set of four squares in the multiplication table that form the corners of a rectangle (with horizontal and vertical sides).

Patterns at the corners of rectangles

If the corners of the rectangle are defined by rows a and c and by columns b and d, then their squares contain the products shown below.



The products of opposite corners are $(ab)(cd)$ and $(ad)(cb)$, which are equal by the commutative and associative properties of multiplication.

Some students may recognize that corner squares always form equivalent fractions.

$$\frac{ab}{cb} = \frac{ad}{cd} \quad \text{or} \quad \frac{ab}{ad} = \frac{cb}{cd}$$

Therefore, this rectangle pattern is simply the familiar cross-product relationship for equivalent fractions!

You could also use a base square to carry out a proof. Since the rectangles do not necessarily have a "center square," you might choose one corner (say UL) as the base square, meaning that UL is in row a and column b. Suppose that UR is w squares to the right of UL, and LL is l squares below UL. Then the algebraic expressions for each corner are:

Upper left (UL)	ab
Upper right (UR)	$a(b + w)$
Lower left (LL)	$(a + l)b$
Lower right (LR)	$(a + l)(b + w)$

UL • LR =

$[ab][(a + l)(b + w)] =$

$[a(b + w)][(a + l)b] =$ (commutative and associative properties of multiplication)

UR • LL

Stage 2

In Stage 2, students use multiplication tables to explore well-known algebraic patterns involving square numbers and differences of squares. They investigate the results of extending these patterns beyond whole numbers by using a multiplication table containing fractions and mixed numbers (or decimals).

Handouts #1 and #2 at the end of this project provide unshaded multiplication tables that students may find helpful in exploring the questions and extending the investigation.

What students should know

- Describe patterns using algebraic expressions.
- Simplify polynomials by combining like terms
- Multiply simple polynomials, including binomials by binomials.

What students will learn

- Recognize, extend, and describe complex patterns.
- Use algebraic processes to prove conjectures.
- Deepen knowledge of known algebraic relationships and discover new ones.

Problem #3

·	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

·	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Directions

- Describe patterns within and between the bold-outlined squares and the other shaded squares.
- Generalize any of the patterns that you can.
- Try to prove the pattern(s) algebraically. If this is too difficult, use words or diagrams to explain what causes the patterns.

Solutions for #3

Some key patterns in the left table

- The numbers along the main diagonal are square numbers.
- The number that is 1 square above and to the right of each main-diagonal number is always 1 less than the number on the main diagonal.

Algebraic proofs of the key patterns in the left table

- The row and column numbers, a and b, are equal for each number on the main diagonal. Suppose that $a = b = n$. Then the main diagonal number is
$$a \cdot b = n \cdot n = n^2.$$
In other words, it is a square number.
- The number that is 1 square above and to this right of the main diagonal number belongs to row $n - 1$ and column $n + 1$. An algebraic expression for its value is
$$(n-1)(n+1) =$$
$$n^2 + n - n - 1 =$$
$$n^2 - 1,$$
which shows that it is *always* 1 less than the corresponding main-diagonal value.

A key pattern in the right table

The number that is 2 squares above and to the right of each main-diagonal number is 4 less than the number on the main diagonal.

Algebraic proof of the key pattern in the right table

The number that is 2 squares above and to this right of the main diagonal number belongs to row $n - 2$ and column $n + 2$. An algebraic expression for its value is
$$(n-2)(n+2) =$$
$$n^2 + 2n - 2n - 4 =$$
$$n^2 - 4,$$
which shows that it is *always* 4 less than the corresponding main-diagonal value.

A generalization and a proof

The number that is k squares above and to the right of the main diagonal number belongs to row $n - k$ and column $n + k$. An algebraic expression for its value is
$$(n-k)(n+k) =$$
$$n^2 + kn - kn - k^2 =$$
$$n^2 - k^2,$$

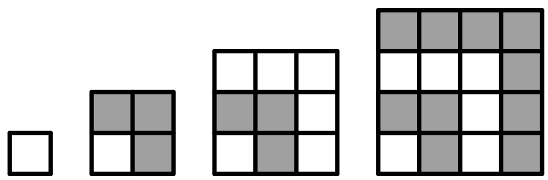
which shows that it is *always* k^2 less than the corresponding main-diagonal value.

A pattern within the main diagonal

The square numbers on the main diagonal are equal to sums of consecutive odd numbers. For example,

$$1^2 = 1 = \mathbf{1}$$
$$2^2 = 4 = 1 + \mathbf{3}$$
$$3^2 = 9 = 1 + 3 + \mathbf{5}$$
$$4^2 = 16 = 1 + 3 + 5 + \mathbf{7}$$
etc.

You can illustrate and justify the pattern with pictures.



Another option is to examine differences between the consecutive square numbers n^2 and $(n+1)^2$.

$$(n+1)^2 - n^2 =$$
$$(n+1)(n+1) - n^2 =$$
$$n^2 + n + n + 1 - n^2 =$$
$$2n + 1$$

Because n is a whole number, $2n$ is an even number. Therefore, $2n + 1$ must be an odd number.

To show that consecutive differences are *consecutive* odd numbers, you can examine the difference between the next pair of square numbers, $(n+1)^2$ and $(n+2)^2$.

$$(n+2)^2 - (n+1)^2 =$$
$$(n+2)(n+2) - (n+1)(n+1) =$$
$$n^2 + 2n + 2n + 4 - (n^2 + n + n + 1) =$$
$$n^2 + 4n + 4 - (n^2 + 2n + 1) =$$
$$n^2 + 4n + 4 - n^2 - 2n - 1 =$$
$$2n + 3$$

$2n + 3$ is 2 greater than $2n + 1$. Therefore, it is next in the sequence of consecutive odd numbers after $2n + 1$.

There are many other possible patterns and methods of proving them.

Problem #4

·	0		1		2		3		4	
0	0		0		0		0		0	
1	0		1		2		3		4	
2	0		2		4		6		8	
3	0		3		6		9		12	
4	0		4		8		12		16	

·	0		1		2		3		4	
0	0		0		0		0		0	
1	0		1		2		3		4	
2	0		2		4		6		8	
3	0		3		6		9		12	
4	0		4		8		12		16	

Directions

- Find the values that belong the empty shaded squares.
- Use your values to compare and contrast patterns in these tables to the patterns in Problems #2 and #3.
- Use algebra to prove as many of your comparisons and discoveries as you can.

Diving Deeper

- Experiment with extending the multiplication table upward and to the left into negative numbers. Look for similar types of patterns.

Solutions for #4

The full table (with shading from the left table)

Students need not complete the entire table, but I include it here for reference.

·	**0**	$\frac{1}{2}$	**1**	$1\frac{1}{2}$	**2**	$2\frac{1}{2}$	**3**	$3\frac{1}{2}$	**4**	$4\frac{1}{2}$
0	0	0	0	0	0	0	0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$
1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$
$1\frac{1}{2}$	0	$\frac{3}{4}$	$1\frac{1}{2}$	$2\frac{1}{4}$	3	$3\frac{3}{4}$	$4\frac{1}{2}$	$5\frac{1}{4}$	6	$6\frac{3}{4}$
2	0	1	2	3	4	5	6	7	8	9
$2\frac{1}{2}$	0	$1\frac{1}{4}$	$2\frac{1}{2}$	$3\frac{3}{4}$	5	$6\frac{1}{4}$	$7\frac{1}{2}$	$8\frac{3}{4}$	10	$11\frac{1}{4}$
3	0	$1\frac{1}{2}$	3	$4\frac{1}{2}$	6	$7\frac{1}{2}$	9	$10\frac{1}{2}$	12	$13\frac{1}{2}$
$3\frac{1}{2}$	0	$1\frac{3}{4}$	$3\frac{1}{2}$	$5\frac{1}{4}$	7	$8\frac{3}{4}$	$10\frac{1}{2}$	$12\frac{1}{4}$	14	$15\frac{3}{4}$
4	0	2	4	6	8	10	12	14	16	18
$4\frac{1}{2}$	0	$2\frac{1}{4}$	$4\frac{1}{2}$	$6\frac{3}{4}$	9	$11\frac{1}{4}$	$13\frac{1}{2}$	$15\frac{3}{4}$	18	$20\frac{1}{4}$

Patterns of sums in the left table

The patterns of sums are similar to those in the original table in Problem #3. The "diagonal" numbers are now $\frac{1}{2}$ (instead of 2) greater or less than twice the center number. The sum of all four diagonal numbers is still four times the center number.

UL + LR is always $\frac{1}{2}$ greater than twice C.

UR + LL is always $\frac{1}{2}$ less than twice C.

$UL + LR = 2C + \frac{1}{2}$

$UR + LL = 2C - \frac{1}{2}$

$UL + UR + LL + LR = 4C$

Proofs of the patterns of sums in the left table

UL + LR =

$\left(a - \frac{1}{2}\right)\left(b - \frac{1}{2}\right) + \left(a + \frac{1}{2}\right)\left(b + \frac{1}{2}\right) =$

$\left(ab - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{4}\right) + \left(ab + \frac{1}{2}a + \frac{1}{2}b + \frac{1}{4}\right) =$

$2ab + \frac{1}{2} =$

2C + $\frac{1}{2}$

UR + LL =

$\left(a - \frac{1}{2}\right)\left(b + \frac{1}{2}\right) + \left(a + \frac{1}{2}\right)\left(b - \frac{1}{2}\right) =$

$\left(ab + \frac{1}{2}a - \frac{1}{2}b - \frac{1}{4}\right) + \left(ab - \frac{1}{2}a + \frac{1}{2}b - \frac{1}{4}\right) =$

$2ab - \frac{1}{2} =$

2C – $\frac{1}{2}$

UL + LR + UR + LL = (2C + $\frac{1}{2}$) + (2C – $\frac{1}{2}$) = 4C

Patterns of products in the left table

The patterns of products in the left table are the same as those in Problem #3: the products of numbers in opposite corner squares are equal.

$$\frac{1}{4} \cdot 2\frac{1}{4} = \frac{9}{16} = \frac{3}{4} \cdot \frac{3}{4}$$

$$3 \cdot 8 = 24 = 4 \cdot 6$$

$$7\frac{1}{2} \cdot 14 = 105 = 10\frac{1}{2} \cdot 10$$

The proof of this fact is identical to the (first) corresponding proof in Problem #3.

The full table (with shading from the right table)

Again, students need not complete the entire table.

·	**0**	$\frac{1}{2}$	**1**	$1\frac{1}{2}$	**2**	$2\frac{1}{2}$	**3**	$3\frac{1}{2}$	**4**	$4\frac{1}{2}$
0	0	0	0	0	0	0	0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$
1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$
$1\frac{1}{2}$	0	$\frac{3}{4}$	$1\frac{1}{2}$	$2\frac{1}{4}$	3	$3\frac{3}{4}$	$4\frac{1}{2}$	$5\frac{1}{4}$	6	$6\frac{3}{4}$
2	0	1	2	3	4	5	6	7	8	9
$2\frac{1}{2}$	0	$1\frac{1}{4}$	$2\frac{1}{2}$	$3\frac{3}{4}$	5	$6\frac{1}{4}$	$7\frac{1}{2}$	$8\frac{3}{4}$	10	$11\frac{1}{4}$
3	0	$1\frac{1}{2}$	3	$4\frac{1}{2}$	6	$7\frac{1}{2}$	9	$10\frac{1}{2}$	12	$13\frac{1}{2}$
$3\frac{1}{2}$	0	$1\frac{3}{4}$	$3\frac{1}{2}$	$5\frac{1}{4}$	7	$8\frac{3}{4}$	$10\frac{1}{2}$	$12\frac{1}{4}$	14	$15\frac{3}{4}$
4	0	2	4	6	8	10	12	14	16	18
$4\frac{1}{2}$	0	$2\frac{1}{4}$	$4\frac{1}{2}$	$6\frac{3}{4}$	9	$11\frac{1}{4}$	$13\frac{1}{2}$	$15\frac{3}{4}$	18	$20\frac{1}{4}$

Patterns in the right table

- The numbers along the main diagonal are squares of the multiples of $\frac{1}{2}$.

$$0^2 = 0 \qquad \left(\frac{1}{2}\right)^2 = \frac{1}{4} \qquad 1^2 = 1 \qquad \left(1\frac{1}{2}\right)^2 = 2\frac{1}{4} \qquad \text{etc.}$$

- The number that is 1 square above and to the right of each main-diagonal number is always $\frac{1}{4}$ less than the number on the main diagonal.
- In general, the number that is k squares above and to the right of the main diagonal number is $\left(\frac{k}{2}\right)^2$ or $\frac{k^2}{4}$ less than the number on the main diagonal, because

$$\left(n - \tfrac{k}{2}\right)\left(n + \tfrac{k}{2}\right) =$$
$$n^2 + \tfrac{k}{2}n - \tfrac{k}{2}n - \tfrac{k^2}{4} =$$
$$n^2 - \tfrac{k^2}{4}.$$

Students may have many other observations, generalizations, and proofs.

Stage 3

In Stage 3, students investigate a pair of complex patterns that lead to a beautiful and surprising connection between square and cube numbers.

What students should know

- Describe patterns using algebraic expressions.
- Simplify polynomials by combining like terms
- Multiply simple polynomials, including binomials by binomials.
- Think of complex expressions as a single quantity.

What students will learn

- Recognize, extend, and describe complex patterns.
- Use algebraic processes to prove conjectures.
- Deepen knowledge of known algebraic relationships and discover new ones.
- Use the distributive property in reverse (to factor complex expressions).

Problem #5

·	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Directions

- Find a pattern(s) involving sums in the different regions (shaded and unshaded).
- Explain what causes the pattern(s). Use algebra when and if it makes your explanation more convincing or clearer.

Solutions for #5

The key pattern

The sums in the shaded "L" regions are successive perfect cubes.

$$1 = 1 = 1^3$$
$$2 + 4 + 2 = 8 = 2^3$$
$$3 + 6 + 9 + 6 + 3 = 27 = 3^3$$
$$4 + 8 + 12 + 16 + 12 + 8 + 4 = 64 = 4^3$$
$$5 + 10 + 15 + 20 + 25 + 20 + 15 + 10 + 5 = 125 = 5^3$$

etc.

The sums are the cubes of the row/column numbers for each "L."

The cause of the pattern

First, notice that each corner number is the square of the row/column number, n.

You can regroup each sum into n groups of n^2, which equals $n^2 \cdot n = n^3$.

$$2 + 4 + 2 =$$
$$4 + (2 + 2) =$$
$$4 + 4 =$$
$$2^2 + 2^2 =$$
$$2^2 \cdot 2 = 2^3$$

$$3 + 6 + 9 + 6 + 3 =$$
$$9 + (3 + 6) + (6 + 3) =$$
$$9 + 9 + 9 =$$
$$3^2 + 3^2 + 3^2 =$$
$$3^2 \cdot 3 = 3^3$$

$$4 + 8 + 12 + 16 + 12 + 8 + 4 =$$
$$16 + (4 + 12) + (12 + 4) + (8 + 8) =$$
$$16 + 16 + 16 + 16 =$$
$$4^2 + 4^2 + 4^2 + 4^2 =$$
$$4^2 \cdot 4 = 4^3$$

$$5 + 10 + 15 + 20 + 25 + 20 + 15 + 10 + 5 =$$
$$25 + (5 + 20) + (10 + 15) + (20 + 5) + (15 + 10) =$$
$$25 + 25 + 25 + 25 + 25 =$$
$$5^2 + 5^2 + 5^2 + 5^2 + 5^2 =$$
$$5^2 \cdot 5 = 5^3$$

etc.

In general, when n is odd, there is one copy of n^2 in the corner and $\frac{n-1}{2}$ pairs that have a sum of n^2 in each of the right and bottom pieces of the "L," giving total of

$$1+\frac{n-1}{2}+\frac{n-1}{2}=1+n-1=n$$

copies of n^2.

When n is even, there is one copy of n^2 in the corner, $\frac{n-2}{2}$ pairs that have a sum of n^2 within each of the right and bottom pieces of the "L," and one more pair that contains the middle number ($\frac{n^2}{2}$) from each piece of the "L," giving a total of

$$1+\frac{n-2}{2}+\frac{n-2}{2}+1=1+n-2+1=n$$

copies of n^2.

In order to understand why each of these pairs will *always* have a sum of n^2, it helps to look closely at the list of the multiples of n that occurs in each row (or column) of the "L" (excluding the corner square).

$$n(1) \quad n(2) \quad n(3) \quad \cdots \quad n(k) \quad \cdots \quad n(n-k) \quad \cdots \quad n(n-3) \quad n(n-2) \quad n(n-1)$$

$$n(1)+\ n(n-1)=n+n^2-n=n^2$$
$$n(2)+n(n-2)=2n+n^2-2n=n^2$$
$$n(3)+n(n-3)=3n+n^2-3n=n^2$$

and, in general,

$$n(k)+n(n-k)=kn+n^2-kn=n^2.$$

Notice that the list clearly contains $n-1$ items. If n is odd, then $n-1$ is even, and the list must contain $\frac{n-1}{2}$ pairs. If n is even, then $n-1$ is odd, and there will be one item remaining after the numbers are paired off (the middle number). If you remove this number from the list, $n-2$ numbers remain, and you have $\frac{n-2}{2}$ pairs. These facts provide more detailed support for the observations in the first two paragraphs on this page.

Problem #6

·	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

·	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

·	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Directions

- Find a pattern involving sums in the bold-outlined regions.
- Use algebra to prove your pattern.
- Explain why (or prove that)

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = (1 + 2 + 3 + \cdots + k)^2.$$

Solutions for #6

The key pattern involving sums in the bold-outlined regions

The sums are

$$1 = 1^2$$

$$1 + 2 + 2 + 4 = 9$$
$$9 = 3^2 = (1 + 2)^2$$

$$1 + 2 + 3 + 2 + 4 + 6 + 3 + 6 + 9 = 36$$
$$36 = 6^2 = (1 + 2 + 3)^2$$

$$1 + 2 + 3 + 4 + 2 + 4 + 6 + 8 + 3 + 6 + 9 + 12 + 4 + 8 + 12 + 16 = 100$$
$$100 = 10^2 = (1 + 2 + 3 + 4)^2$$

etc.

Each sum is a perfect square:

$$1, 9, 36, 100, 225, \text{ etc.}$$

The bases of the exponential expressions are

$$1, 3, 6, 10, 15, \text{ etc.},$$

which are the *triangular numbers*. Triangular numbers result from the pattern

$$1$$
$$1 + 2$$
$$1 + 2 + 3$$
$$1 + 2 + 3 + 4$$
$$1 + 2 + 3 + 4 + 5$$

etc.

Understanding and proving the pattern

In order to understand the pattern, try a few examples focusing on one row at a time.

$$(1 + 2) + (2 + 4) =$$
$$(1 + 2) + 2(1 + 2) =$$
$$(1 + 2)(1 + 2) =$$
$$(1 + 2)^2$$

$$(1 + 2 + 3) + (2 + 4 + 6) + (3 + 6 + 9) =$$
$$(1 + 2 + 3) + 2(1 + 2 + 3) + 3(1 + 2 + 3) =$$
$$(1 + 2 + 3)(1 + 2 + 3) =$$
$$(1 + 2 + 3)^2$$

$$(1+2+3+4)+(2+4+6+8)+(3+6+9+12)+(4+8+12+16)=$$
$$(1+2+3+4)+2(1+2+3+4)+3(1+2+3+4)+4(1+2+3+4)=$$
$$(1+2+3+4)(1+2+3+4)=$$
$$(1+2+3+4)^2$$

etc.

The general k^{th} square has the sum

$$\begin{aligned}
&\ \ 1(1+2+3+\cdots+k)\\
&+2(1+2+3+\cdots+k)\\
&+3(1+2+3+\cdots+k)\\
&\qquad\quad\vdots\\
&+k(1+2+3+\cdots+k)\\
\\
&=(1+2+3+\cdots+k)(1+2+3+\cdots+k)\\
&=(1+2+3+\cdots+k)^2
\end{aligned}$$

In all of the examples and in the general pattern, the expression

$$1+2+3+\cdots+k$$

is a common factor within each term. Near the end of the process, you factor this term out.

Proving that $1^3+2^3+3^3+\cdots+k^3=(1+2+3+\cdots+k)^2$

Imagine the bold-outlined square in the multiplication table whose rows (and columns) run from 1 to k. Consider two ways to find the sum of all numbers in this square.

(1) Using the ideas from Problem #5, add the sums in each of the “Ls.” Since each sum is the cube of its row/column number, the result is

$$1^3+2^3+3^3+\cdots+k^3.$$

(2) Using the ideas from earlier in this problem, add the numbers one row at a time, factoring as needed. The result is

$$(1+2+3+\cdots+k)^2$$

Since the sum must be the same regardless of how you calculate it, the two expressions above must be equivalent. That is:

$$1^3+2^3+3^3+\cdots+k^3=(1+2+3+\cdots+k)^2.$$

Handout #1

·	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Handout #2

$\cdot$	**0**	$\frac{1}{2}$	**1**	$1\frac{1}{2}$	**2**	$2\frac{1}{2}$	**3**	$3\frac{1}{2}$	**4**	$4\frac{1}{2}$
0	0	0	0	0	0	0	0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$
1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$
$1\frac{1}{2}$	0	$\frac{3}{4}$	$1\frac{1}{2}$	$2\frac{1}{4}$	3	$3\frac{3}{4}$	$4\frac{1}{2}$	$5\frac{1}{4}$	6	$6\frac{3}{4}$
2	0	1	2	3	4	5	6	7	8	9
$2\frac{1}{2}$	0	$1\frac{1}{4}$	$2\frac{1}{2}$	$3\frac{3}{4}$	5	$6\frac{1}{4}$	$7\frac{1}{2}$	$8\frac{3}{4}$	10	$11\frac{1}{4}$
3	0	$1\frac{1}{2}$	3	$4\frac{1}{2}$	6	$7\frac{1}{2}$	9	$10\frac{1}{2}$	12	$13\frac{1}{2}$
$3\frac{1}{2}$	0	$1\frac{3}{4}$	$3\frac{1}{2}$	$5\frac{1}{4}$	7	$8\frac{3}{4}$	$10\frac{1}{2}$	$12\frac{1}{4}$	14	$15\frac{3}{4}$
4	0	2	4	6	8	10	12	14	16	18
$4\frac{1}{2}$	0	$2\frac{1}{4}$	$4\frac{1}{2}$	$6\frac{3}{4}$	9	$11\frac{1}{4}$	$13\frac{1}{2}$	$15\frac{3}{4}$	18	$20\frac{1}{4}$