Deep Algebra Projects: Pre-Algebra/Algebra 1
Integer Interpreter

Topics
- Addition and subtraction with negative and positive numbers (Stage 1)
- Multiplication and division with negative and positive numbers (Stages 2 and 3)
- Linear relationships and rates of change
- Representing relationships with graphs, tables, formulas, and real-world contexts
- Algebraic expressions and equivalence

The Integer Interpreter activity helps students develop a deeper understanding of the *meanings* and *properties* of the four major operations in the context of negative and positive numbers. The title may be a little misleading; students will also work with *fractional* positive and negative values (which are not integers).

This Extension Project is a little different than most others. The stages may not increase in difficulty in the same way as usual. Each stage does carry the concepts to the next level (so you should probably do them in order), but some students may find the problems in Stage 1 to be just as challenging as those later in the project. When you choose problems to assign, you may want to focus more on the concepts you want to teach than on the level of challenge.

You can use this activity *before or after* students learn rules for adding and subtracting with negative and positive numbers. Using it before they know the rules will make for a more challenging experience but will also force kids to think more deeply about the ideas, because they will not have a set of procedures to fall back on. They may *discover* traditional rules in the process! If you use it after they have learned rules, they will develop a deeper understanding of the rules, but you may have to press them to go beyond comfortable procedures into new ways of thinking about the ideas.

Suggestion: Before you give students the handout for each problem, show them just the image at the top of the page, and ask them to notice and wonder as much about it as they can. Use their ideas to help create the directions for the problem. This may mean adding to or modifying the directions in some cases!

Stage 1

Note: Do not distribute Problems #2 and #3 until students have completed Problem #1.

In Stage 1, students explore addition and subtraction of positive and negative numbers visually, often without knowing actual values for the numbers that they are adding and subtracting. This helps them learn to think deeply and abstractly about *relationships* rather than focusing on rules for calculating. Often, the discoveries that they make can be expressed geometrically (using measurements on a number line) or algebraically.

Most students will probably be more successful if they start by choosing specific numbers in order to get a feel for what is happening. (You may need to suggest this if they don't hit upon the idea themselves.) However, after they have experimented with numbers for a while, encourage them to try thinking without numbers to see how far they can get. Often, it will be possible to answer questions without knowing values!

A caution about Problem #1: The first part of the problem (and a piece of the second part) has no solution. It is not meant to trick students but to make them think more deeply. The more often that students experience problems that have no solution, the more they come to expect and accept them as a natural part of doing mathematics.

What students should know*
- Understand how positive and negative numbers are ordered on a number line.
- Be aware of multiple meanings and models for subtraction.
- Understand that "–" can mean minus, negative, or opposite depending on the circumstances.
- Write and interpret simple algebraic expressions.

What students will learn
- Develop a deeper understanding of addition and subtraction by solving challenging problems and exploring properties of the operations in situations that involve negative and positive numbers.

*Note: Students may use this activity with or without knowing rules for adding and subtracting positive and negative numbers. See the Introduction to the activity for more information.

Problem #1
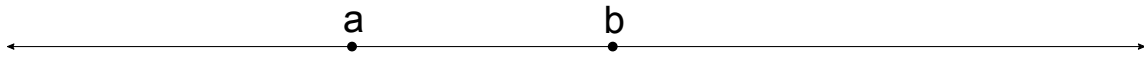
$$\xleftarrow{\hspace{2cm}\underset{a}{\bullet}\hspace{2cm}\underset{b}{\bullet}\hspace{4cm}}\rightarrow$$

a   b

Directions

- The numbers a and b are shown. Show a + b, a − b, and b − a. Explain.
- Decide if each expression is positive or negative. Explain.

Conversation Starters for #1
What do you notice? What do you wonder?

I notice that I don't know anything about the size of a or b.

I notice that I don't know anything about the distance between a and b.

I notice that I don't know whether a and b are positive or negative.

I wonder how I can find a sum or difference without knowing these things.

I wonder if it would help to start by choosing a variety of numbers for a and b.

I wonder what is the least amount of additional information I would need in order to find the positions of the expressions.

I notice that b is greater than a.

I notice that there are many ways to think of subtraction.
> You can think of it as "take away," "how much more," "counting up," "what I need to add," "the opposite of addition," etc. Different meanings may work better for you in different situations.

I notice that I can show b – a as the length of a segment but not as a point on the line (because I don't know where the segment starts).

I wonder if b – a is *always* positive when b > a (even when one or both of the numbers are negative).

I notice that I need to know where 0 is in order to locate the expressions.

I notice that, even when I know where 0 is, I still don't know how far a is from b (as a number).

I notice that (once I know where 0 is), the scale on the line doesn't affect the position of the expressions.

Solutions for #1

Showing the expressions

Students should discover (after some thinking, which may include testing different values for a and b), that it is impossible to find a place on the line for any of the expressions. However, they could do it if they knew the where 0 was—even without knowing actual values for a and b! (See Problem #2.)

Some students may notice that, although you cannot show $a - b$ or $b - a$ as points on the line, you can represent them as the *distance* between a and b, except that $a - b$ is negative, because b is greater than a. In fact, $a - b$ is the opposite of $b - a$.

Deciding if the expressions are positive or negative

$a + b$ may be positive or negative.

If 0 is to the left of a and b, then $a + b$ is positive (and to the right of a and b).

If 0 is to the right of a and b, then $a + b$ is negative (and to the left of a and b).

If 0 is between a and b, then $a + b$ is between a and b, but could be either positive or negative.

$b - a$ is positive, because $b > a$ (b is to the right of a on the number line). This is true regardless of whether a and b themselves are positive or negative.

$a - b$ is negative for the same reason.

Problem #2
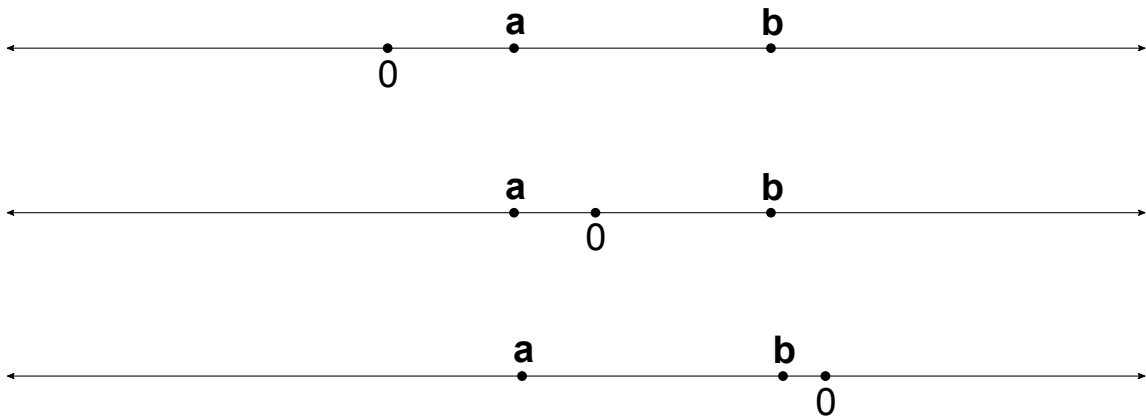
Directions

- Show a + b, a − b, and b − a on each number line. Explain.
- Explain how moving 0 affects the positions of a + b, a − b and b − a.
- Write a real-world story for b − a that makes sense for any of the three number lines.

Diving Deeper

Explain *why* the points move as they do when 0 moves.

Conversation Starters for #2
What do you notice? What do you wonder?

I notice that a and b stay where they are, but 0 moves.

I notice that I can locate a + b, a − b, and b − a on the line, even though I don't know specific numbers for a and b.

I notice that, although a and b *appear* not to move, their numbers would change when 0 moves.

I notice that moving 0 to the different places on the line affects whether a and b are positive or negative.

I wonder if it would help to start by choosing a variety of numbers for a and b.

I notice that when 0 moves to the right, the numbers for a and b each decrease by the same amount that 0 moves.

I notice patterns in how a + b, a − b, and b − a appear to move when 0 moves.

I wonder what causes the patterns.

I notice that 0 is always exactly halfway between a − b and b − a.

I wonder why this happens.

I wonder if I could see anything interesting by lining up the 0 on each line.

I wonder if it would help to think of subtraction as a distance instead of as taking away.
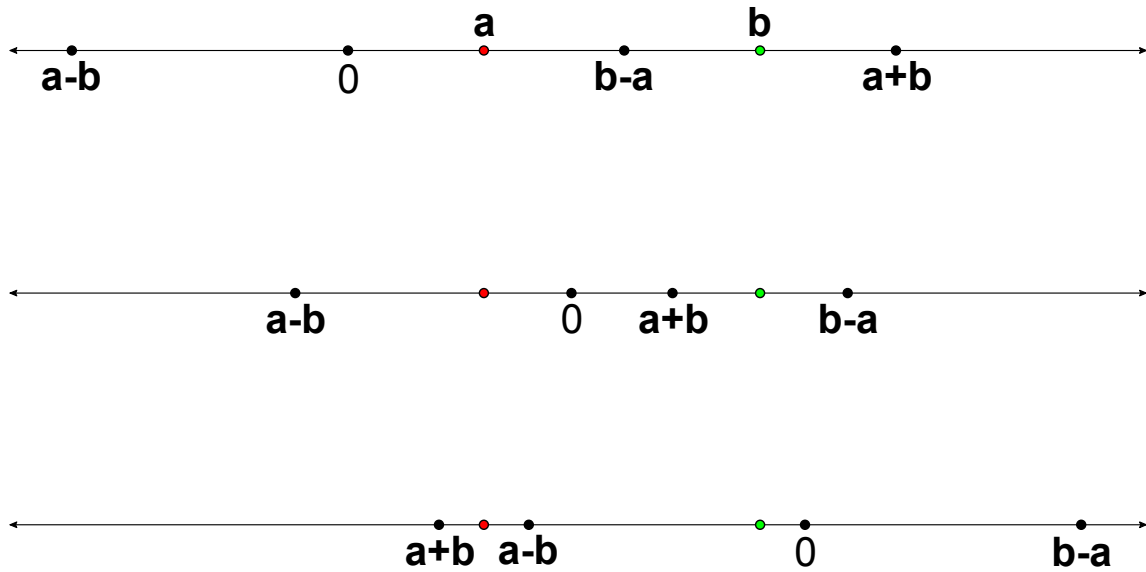
I notice that when I know how a and b change individually, I can predict how a + b, a − b, and b − a will change.

I notice that when a and b change by the same amount (and in the same direction), their difference does not change.

Solutions for #2

*Note*: In the solutions, the red point is a and the green point is b.

Showing the three expressions

The effects of moving 0 to the right by x

Moving 0 to the right by the distance x also shifts a – b and b – a to the right by x *relative to a and b*. It moves a + b to the left by x—again, *relative to a and b*.

The actual *values* of a – b and b – a remain unchanged, while the value of a + b decreases by 2x. You can see this more easily if you align the zeros.
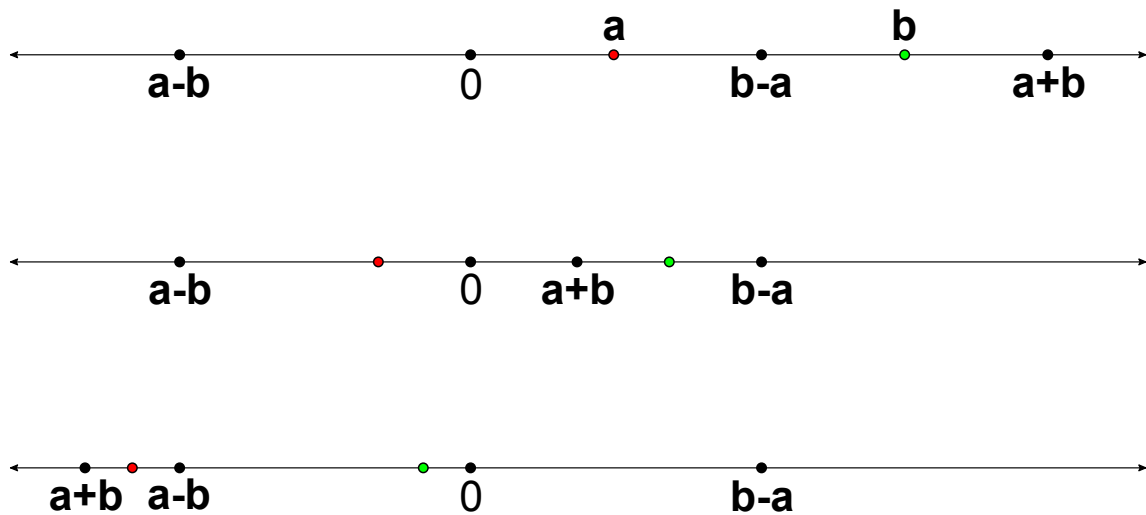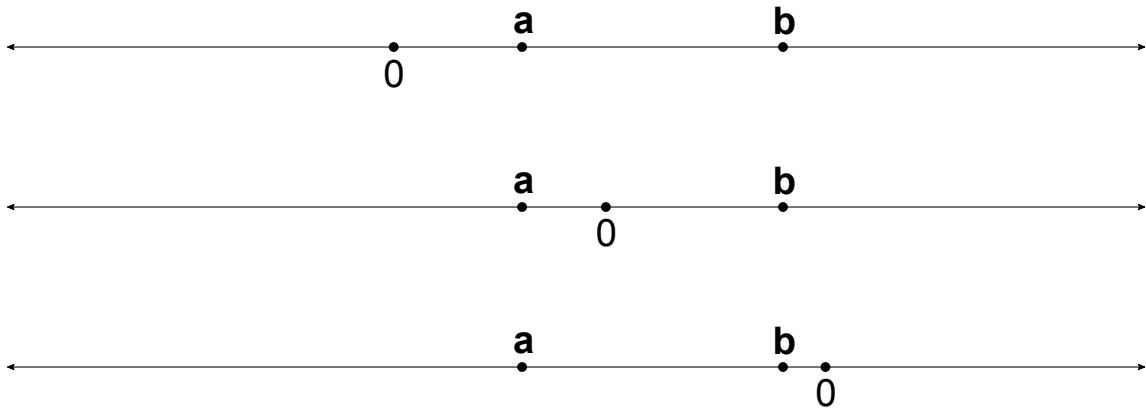

A sample story

The temperature at noon is a°F. At 3pm, it is b°F. By how many degrees did the temperature increase in that time? (A negative answer reflects a decrease.)

Note: Stories that use the "how much more" (comparison) meaning of subtraction may work more easily for all three lines than stories using a "take away" meaning.

Problem #3

Directions

- Show –(a + b), -a + b, -(a – b), and –a – b on each number line. Explain.
- Use your answers above to identify any equivalent expressions. Explain.

Diving Deeper

Explain how moving 0 affects the values of the expressions.

Conversation Starters for #3
What do you notice? What do you wonder?

I notice that the number lines in this problem are the same as the ones in Problem #2.

I notice that the algebraic expressions in this problem are more complex that the ones in Problem #2.

I notice patterns in the algebraic expressions.

I wonder if these patterns will show up in some way on the number line.

I notice that -a is not necessarily a negative number, because it is the opposite of whatever a is.

I notice that in two of the expressions, you add or subtract before you take the opposite. In the other two expressions, you take the opposite first.

I wonder if it would help to start by choosing a variety of numbers for a and b.

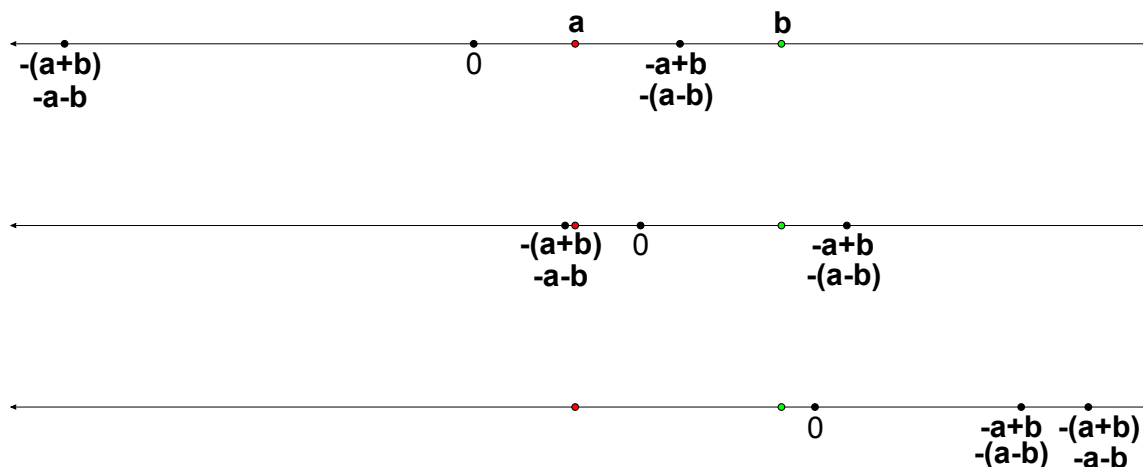*I notice* that different expressions sometimes end up in the same position on the line.

I wonder what else this tells me about these expressions.

I notice a connection to the distributive property.

I notice some common-sense reasons that some of the expressions are equivalent.

Solutions for #3

Showing the expressions on the lines

Students will have many procedures for locating the expressions. Some may substitute reasonable numbers for a and b and then calculate. Encourage them eventually to search for a general procedure (that does not depend on choosing specific numbers).

A sample thinking process (top number line, leftmost expressions)

   -(a + b):

- Join the length a (the distance between 0 and a) to the length b to form a + b.
- Place a point at this distance to the right of 0 to mark the sum.
- Find the mirror image of this point across 0 to take the opposite of this sum.

   -a – b:

- Find the mirror image across 0 of the point at a to take the opposite of a.
- Find the distance between -a and b.
- Place a point at this distance to the *left* of 0. Place it to the *left* because the answer is negative; you are subtracting a larger number (b) from a smaller one (-a).

Each expression describes a different process, but they lead to the same point on the line!

Note: Above, I thought of subtraction using the idea of distance. Students will have other approaches.

Equivalent expressions

-(a + b) is equivalent to -a − b.

-(a − b) is equivalent to -a + b.

In each case, the two expressions are at the same location on each number line, even though they represent different processes.

Some students may have had experience using the distributive property with negative numbers. If so, ask them to explain the equivalent expressions by multiplying a + b and a − b by -1. If not, they may still recognize patterns in the expressions that help them discover rules for taking the opposite of a sum or a difference.

Either way, encourage them also to interpret the expressions directly (as shown in the example on the previous page).

Stage 2

In Stage 2, students use graphs, tables, formulas, and real-world contexts to explore multiplication of negative and positive numbers. They can use the problems with or without knowing rules for these calculations. If they know the rules, they will learn to understand them more deeply. If they do not, the activity will be more challenging, but they will have an opportunity to discover the rules and understand more about the *why* behind them.

In Problem #4, students are asked to identify a rate of change in a linear relationship. If they have already learned the concept and vocabulary for *slope*, you might choose to rephrase the directions using this language.

What students should know
- Understand that a *linear* relationship has a constant rate of change.
- Identify the rate of change of a linear relationship.
- Use formulas, tables, and graphs to represent linear relationships.

What students will learn
- Extend understanding of linear relationships to incorporate negative rates of change.
- Use linear relationships to explore and understand the rules for multiplying negative and positive numbers.
- Understand that you can sometimes choose the meaning of the "0" point in a relationship that involves negative values.

Problem #4
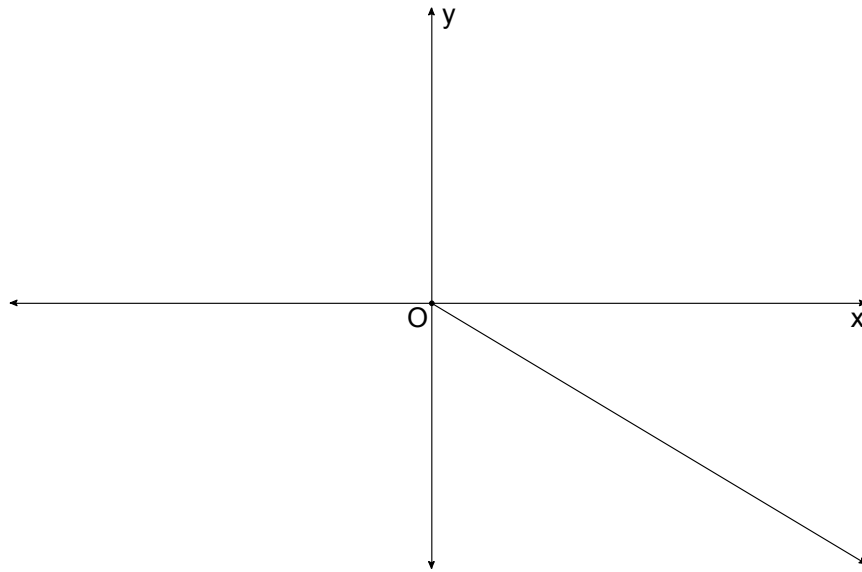
Directions

- Create scales on the axes, and describe a real-world scenario to fit the graph.
- Identify the rate of change in your scenario. Explain how it relates to the graph.
- Create a table and a formula for your scenario.
- Write three multiplication equations for this situation.
- Explain how your equations fit your scenario.

Conversation Starters for #4
What do you notice? What do you wonder?

Suggestion: As mentioned in the Introduction to this project, show the image to students before they see the directions. Let them notice and wonder as much as they can with this first. They may be able to predict or create many of the directions on their own. They may even come up with their own ideas for things to explore.

I notice that the graph is a ray stays to the right of the *x*-axis

I notice that ray is dropping from left to right.

I notice that the ray starts at (0, 0) and descends into negative numbers.

I notice that every point on the ray except (0, 0) has a positive *x*-coordinate and a negative *y*-coordinate.

I notice that the rate of change depends on how I choose the scales on the axes.

I notice that the output is going down while the input goes up.

I notice that the rate of change will be negative no matter how I choose my scales.

I notice that my real-world scenario will have to involve negative numbers.

I notice that it is challenging to create a realistic scenario that includes (0, 0).

I wonder if it is okay for me to decide what 0 stands for in my real-world scenario.

I wonder if my formula will have to include a negative number.

I notice that the rate of change is part of my equation.

I notice that my formula involves multiplying with negative numbers.

I wonder what would happen if I extended the ray to form a line.

Solutions for #4

Creating scales and a scenario

If students use the same unit of length on the *x*-axis and the *y*-axis, the point with coordinates (5, -3) will be on the ray. A sample story:

> Hikers who have been descending a mountain at a steady rate of 3 feet every 5 seconds reach a fork in the trail. If they keep descending at the same rate, the graph describes their elevation (*y*) relative to the fork in feet for a given number of seconds (*x*) after they reach the fork.

The rate of change in the scenario

If students use the same unit of length on the *x*- and the *y*-axis, the rate of change will be $-\frac{3}{5}$ (or -0.6). This represents the hikers' rate of *descent* in feet per second.

The rate of change relates to the *steepness* of the graph. A *negative* rate of change appears as a graph that *drops* from left to right, because as time increases, height decreases. The steeper the drop, the faster the hikers are descending.

If students choose different units of length for each axis, the rate of change will depend on their choices, but it is certain to be negative. Use the opportunity to compare the effects of their choices.

A sample table

x	0	1	2	3	4	5	6	7	8	9	10
y	0	-0.6	-1.2	-1.8	-2.4	-3.0	-3.6	-4.2	-4.8	-5.4	-6.0

Sample formula

$$y = -0.6x$$

Sample multiplication equations

-0.6 • 2 = -1.2   -0.6 • 5 = -3   -0.6 • 7 = -4.2

Connecting the equations to the scenario

    Equation 1:

        -0.6 • 2 = -1.2 shows that if the hikers descend at 0.6 feet per second, after 2 seconds, they will be 1.2 feet below the fork.

    Equation 2:

        -0.6 • 5 = -3.0 shows that if the hikers descend at 0.6 feet per second, after 5 seconds, they will be 3 feet below the fork.

    Equation 3:

        -0.6 • 7 = -4.2 shows that if the hikers descend at 0.6 feet per second, after 7 seconds, they will be 4.2 feet below the fork.

Note: Choosing the fork in the trail as a reference point for both the time and the elevation sets the elevation to 0 when the time is 0. In other words, it ensures that the graph contains the origin (0, 0). In situations involving negative numbers, it is often possible to choose reference points for the input and/or output.

Problem #5
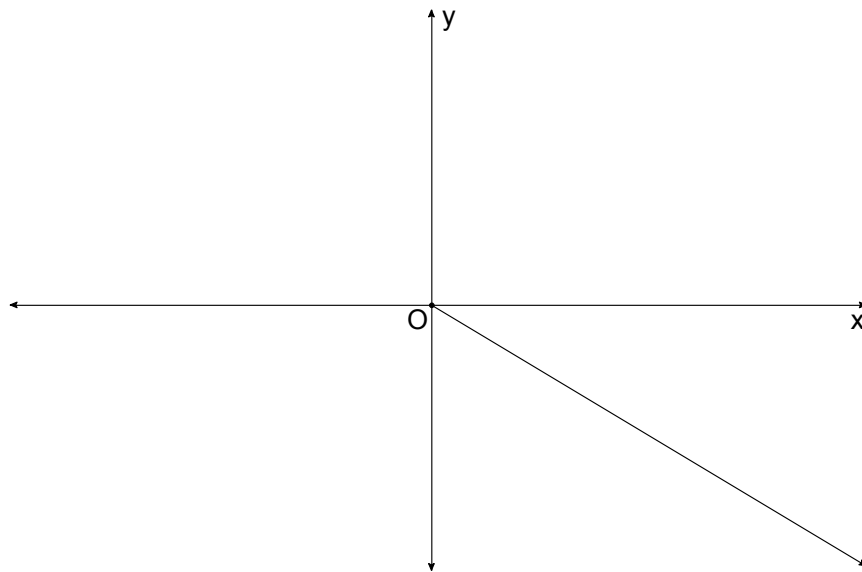
Directions

- Extend the graph to create a line. Then extend the table to match.
- If your scenario from Problem #4 fits the extended graph and table, explain how. If not, create a new one that does, and how explain how it fits.
- Write three multiplication equations from the new part of the graph.
- Explain how your new equations fit your scenario.

Conversation Starters for #5
What do you notice? What do you wonder?

I notice that extending the ray brings in negative inputs.

I notice that the formula does not change.

I notice that the new part of my graph has all negative inputs and positive outputs.

I notice that negative inputs do not make sense in my scenario.
> This may be true for some students. Something to think about: Is it possible to "reset" the meaning of 0 so that the negative inputs do make sense? If not, the student will need to think of a new scenario.

I notice that the new part of the graph has both a negative rate of change and negative inputs.

I notice that this involves multiplying two negative numbers.

*I notice* that if I trace the graph backwards, the outputs increase.

I notice that changing the input to its opposite changes the output to its opposite.

I wonder if multiplying two negative numbers *always* gives a positive answer.
> Of course, this question will come up only for students who have not already been taught rules for multiplying with negative numbers. If it does arise, encourage them to explore similar linear patterns in tables and graphs with a variety of numbers.

Solutions for #5

Extending the graph
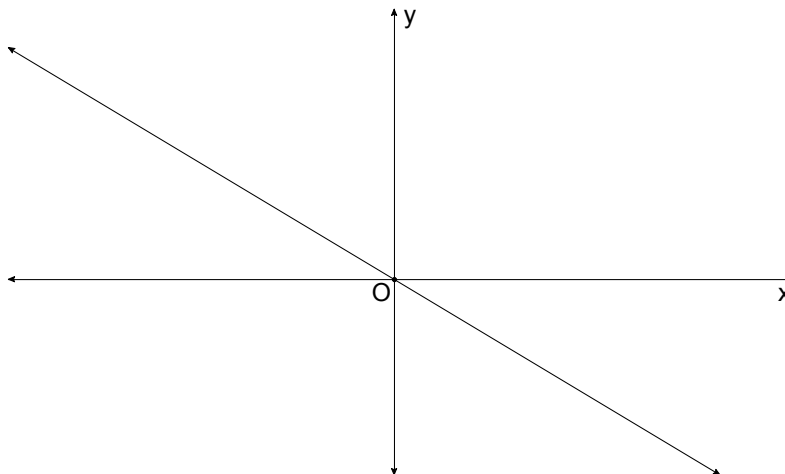
Extending the table (sample)

x	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
y	6.0	5.4	4.8	4.2	3.6	3.0	2.4	1.8	1.2	0.6	0

Connecting the extended graph and table to the scenario

The scenario still fits the extended table and graph, because the hikers were descending at the same rate before reaching the fork in the trail. Negative *x* values represent the time before reaching the fork, and the corresponding *y* values stand for their elevation above the fork before reaching it.

If it does not make sense for a student to extend their scenario into negative values of *x*, they should create a new scenario for which this does make sense.

Multiplication equations for the extended scenario

Note: These equations are still based on the formula $y = -0.6x$, because the hikers are descending at the same rate as before.

$$-0.6 \bullet -2 = 1.2 \qquad -0.6 \bullet -5 = 3 \qquad -0.6 \bullet -7 = 4.2$$

Connecting the equations to the extended scenario

The first number in the equation is still -0.6, because the hikers have been descending at 0.6 feet per second all along. The second number (input) is now negative, because it refers to times *before* the hikers reach the fork. The answer (output) is now positive, because the hikers are *above* the fork before they reach it.

Equation 1:

-0.6 • -2 = 1.2 shows that if the hikers *descend* at 0.6 feet per second, then 2 seconds *before* they reach the fork, they are 1.2 feet *above* it.

Equation 2:

-0.6 •-5 = 3.0 shows that if the hikers *descend* at 0.6 feet per second, then 5 seconds *before* they reach the fork, they are 3 feet *above* it.

Equation 3:

-0.6 • -7 = 4.2 shows that if the hikers *descend* at 0.6 feet per second, then 7 seconds *before* they reach the fork, they are 4.2 feet *above* it.

Note: Students' scenarios offer a chance for them to make sense of the rules for multiplying with negative numbers. In particular, the examples above show that the rule

negative • negative = positive

simply comes from extending a decreasing linear pattern into negative inputs.

Stage 3

In Stage 3, students combine ideas from the first two stages. They explore multiplication with negative and positive numbers on number lines. In the process, they will discover important similarities and differences with the cases of addition and subtraction that they investigated in Stage 1.

Note: Stage 2 did not address division, but the problems in Stage 3 do assume that students understand rules for dividing with positive and negative numbers.

What students should know
- Represent positive and negative numbers (including fractions) on a number line.
- Know and understand rules for multiplying and dividing negative and positive numbers.
- Know and understand rules for multiplying and dividing fractions.

What students will learn
- Develop a deeper understanding of multiplication and division by solving challenging problems and exploring properties of the operations in situations that involve negative and positive numbers.

Problem #6
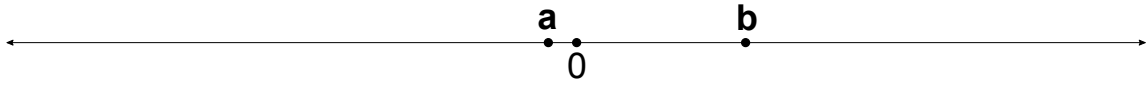
Directions

- The numbers a and b are shown. Show ab, $\dfrac{a}{b}$, and $\dfrac{b}{a}$. Explain.

- Decide if each expression is positive or negative. Explain.

Conversation Starters for #6
What do you notice? What do you wonder?

I notice that a is negative and b is positive.

I notice that a is quite a bit closer to 0 than b is.

I wonder if it would help to choose a variety of values for a and b.

I notice that once I choose a value for a or b, it determines the value for the other point.
> At least, this is true in principle. In practice, you have to *estimate* the number for the second point, because it is not possible to see exactly where it is on the line.

I notice that choosing different values leads to different positions for the expressions.

I wonder if it is possible to make ab land at any point to the left of 0 that I choose by selecting the right value for a or b.

I notice that it is not enough to know where 0 is when I try to locate multiplication and division expressions on the line.

I wonder what other information I need.

I notice that once I know the number for any other point on the line, I know the number for every point on the line.

I notice that is convenient to know the location of 1, because then the distance between 0 and 1 is the basic unit of measurement on the line.

Solutions for #6

Showing the expressions

Students should discover (after some thinking, which may include testing different values for a and b), that it is impossible to find a place on the line for any of the expressions. For multiplication and division, they must know not only where 0 is but where 1 (or some other number) is, because they must establish a unit of length. (See Problem #7.)

Deciding if the expressions are positive or negative

All three expressions are negative, because b is positive and a is negative.

Problem #7
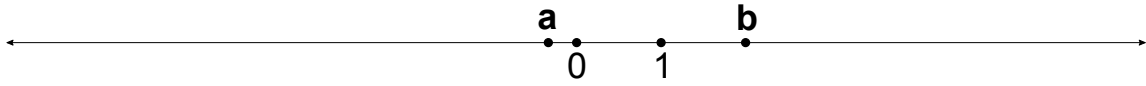
Directions

- Show ab, $\frac{a}{b}$, and $\frac{b}{a}$ on the number line. Explain.

- Imagine gradually sliding 1 toward 0 on the line. Explain how this affects the positions of the three expressions.

Diving Deeper

- Explore the effects of moving 0 and 1 in different ways.
- Create complex expressions such as

$$\frac{-b(a-b)}{-a}$$

locate them on the number line. Rewrite your expressions in equivalent forms and locate them again. Do they land in the same position? Should they? Explain.

Conversation Starters for #7
What do you notice? What do you wonder?

I notice that b is greater than 1, and a is negative.

I notice that b is farther from 1 than a is from 0.

I notice that a and b are a little more than 2 units apart.

I notice that I can determine (or at least estimate) the actual values of a and b.

I notice that sliding 1 towards 0 makes the unit of measurement smaller.

I notice that making the distance between 0 and 1 half as large doubles the values of both a and b.

I notice that making the distance between 0 to 1 *n* times as large multiplies both a and b by the reciprocal of *n*.

I notice that I can make b as large as I want by sliding 1 close enough to 0.

I wonder what would happen if I slid 1 to the right instead.

I wonder what would happen if I slid 1 on top of 0.

I wonder what would happen if I slid 1 past 0 to the left.

I wonder if sliding 0 towards 1 has the same effect as sliding 1 towards 0.

I wonder what would happen if I slid 0 and 1 at the same time, keeping the distance between them the same.
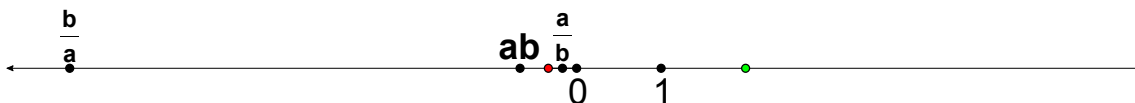
If I did this, *I wonder* what would be the greatest and least possible values of ab (or could I make ab equal any number).

When I know where 0 and 1 are, *I wonder* if I can find the position of every possible algebraic expression containing *a* and *b*.

Solutions for #7

Note: Remember that the red point is at a and the green point is at b.

Showing the expressions on the number line

Including 1 on the line determines the values of a and b (assuming you can read them accurately). b appears to be located at 2, and a appears to be at about -1/3. Based on these values,

$$ab = -\frac{2}{3}$$

$$\frac{a}{b} = -\frac{1}{6}$$

$$\frac{b}{a} = -6.$$

The effects of sliding 1 *toward* 0

As 1 gets closer to 0, ab moves farther and farther to the left. Small changes in the position of 1 have greater and greater effects on the position of ab.

$\frac{a}{b}$ and $\frac{b}{q}$ remain unchanged.

Explanation: When 1 moves toward 0, the size of the unit decreases. Because the unit gets smaller, the absolute value of the number associated with each point increases.

For example, if 1 moves halfway toward 0, the unit is half of what it was, which doubles the values associated with a and b. Since both values double, the fractions remain equivalent.

On the other hand, when a and b both double, the value of ab becomes 2 • 2 = 4 times greater than it was originally. Since ab was negative to start with, making it 4 times greater moves it 4 times farther from 0 (to the left).

The closer 1 gets to 0, the greater the effect will be. For instance, if 1 moves 99/100 of the way toward 0, the unit becomes 1/100 of what it was, making the value of each point 100 times greater, which in turn makes the value of ab 100 • 100 = 10000 times greater!