**Deep Algebra Projects: Pre-algebra / Algebra I**
**Formulas, Tables, Stories, and Graphs**

**Topics**

- Connections between different representations of mathematical functions: formulas, tables, contexts, and graphs
- Rates and unit rates
- Linear relations, rates of changes (slope) and starting numbers (*y*-intercepts)
- Equations of lines, intersections of lines (informally)
- Inverse relations
- Connections between mathematical functions and concepts across the middle school math standards

The Formulas, Tables, Stories, and Graphs (FTSG) project involves students in making connections between different ways of representing and interpreting mathematical relationships. All of the problems use a single template. While the structure of the template provides students with a consistent, comfortable framework for displaying the information, it may be limiting at times. Please encourage those who want to use their own paper in lieu of the template to do so.

Each problem provides a different combination of information. While the problems offer enough variety to make students think of connections from many perspectives, there is nothing special about the choices I have made. You may create problems suited to the exact needs of your students simply by filling in selected portions of a blank template (see the page following this introduction).

While one may use the problems in this project to reinforce previously learned algebraic skills, I always use them *before* students learn rules and formulas involving slopes, *y*-intercepts, etc. Learners who understand ratios, rates, and patterns are able to develop their own strategies for finding these quantities and writing equations.

When discussing the problems, we begin with less formal language (*rate of change* and *starting number*), bringing in terms such as *slope* and *y-intercept* only as students begin to recognize the connections between the concepts and the appearance of the graphs. I find that this approach makes the concepts more meaningful when students encounter them in later algebra courses.

Instead of assigning these problems all at once, you may prefer to intersperse them throughout the school year as they relate to other topics you are studying. For example, when you are studying fractions or negative numbers, you may choose (or create) problems that involve fractional or negative slopes. Or when you are studying angles or measurement, you might select problems relevant to those concepts.

Keep in mind that the Solutions I have shown for these problems are just examples. Students may make different choices about

- writing numbers in fraction vs. decimal form
- choosing names for variables
- which values to include in their tables
- whether or not to connect dots in their graphs
- the locations of axes in the grids on the template
- the scales on their axes,
- ways to write formulas, etc.

They will also have many different strategies for completing tables and finding equations, rates of change, and starting numbers. Ask them to compare their various approaches and to talk about their choices and strategies.

Students' stories will probably be more creative and interesting than mine! They should share and compare these as well. (Some of my stories are not particularly realistic, and a few are frankly contrived. See Problem #7 about Ginny's birthday, for example.) They might consider the realism, creativity, or practicality of their stories. They could also talk about how the concepts of *linear* relationships, *rates of change*, and *starting numbers* relate to their stories. For example, is it realistic to assume that the quantities in their stories have a linear relationship? Thinking of real-world quantities that are linearly related can be quite challenging, and more often than not, the relationship is approximate at best.

## FTSG Templates

linear? ________________

rate of change: ________________

starting number: ________________

formula: ________________

story: ________________

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linear? ________________

rate of change: ________________

starting number: ________________

formula: ________________

story: ________________

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## Stage 1

In the Stage 1 problems, students begin exploring linear relationships that have positive rates of change.

**What students should know**

- Plot points in a rectangular coordinate system.
- Have experience describing patterns as rules (verbally) and as formulas.
- Use the language of *inputs* and *outputs* with regard to rules and patterns.
- Have experience working with ratios, rates, and unit rates.

**What students will learn**

- Understand the meanings of *rate of change* and *starting number* in quantitative relationships.
- Understand the meaning of a *linear* relationship.
- Recognize linear relationships in different representations: formulas, tables, stories, and graphs.
- Translate flexibly between formulas, tables, graphs, verbal descriptions, and stories when representing mathematical relationships.

**Notes on specific problems**

- Problem #1

  Students begin exploring connections between formulas, tables, stories, and graphs. Minimal pre-instruction should be needed, though students may have a few questions about the format of their equations, the meaning of *rate of change* and *starting number*, etc. Students may begin work independently, and you may address questions such as these as they arise.

  The problems in Stage 1 introduce students to the concept of *linear* relationships. In each problem, they should strive to recognize linearity in the different representations. In linear relationships:

  The *graph* is a straight line.
  The *table* shows a constant rate of change.
  The *formula* has a format that they will learn to recognize over time. (See the notes on Problem #5.)
  The *stories* reflect quantities that increase or decrease at a steady rate.

In the second example, students will learn to think of the rate of change as a *unit* rate. In other words, the rate of change is the amount that the output ($y$) changes when the input ($x$) *increases by one*.

Students may notice that the two examples are *inverses* of each other. (The input and the output are interchanged.) Encourage them to pay attention to how this affects the formula, graph, and rate of change. The final three problems in Stage 3 address inverse relationships in much more detail.

- Problem #2
  Students explore the effects of changing the *starting number* (which they may later begin to name as the $y$-intercept.) The rates of change are the same within each example in order that students may focus (for instance) on the effects of adding 1 and subtracting 3 in the patterns. Guide them to notice that the graphs are parallel and that one graph is higher than the other. They may also notice that the graph "begins" or crosses the $y$-axis at the starting number.

  In the first example, students may think of placing the $x$-axis higher in the grid so that they can show the point (0, -1) in their graph. They will also need to think about how to incorporate a negative number into their stories.

- Problem #3
  These two examples are similar to those in Problems #1 and #2 except that students begin with graphs instead of tables. They will need to think about how to create a table of values from each graph.

- Problem #4
  Again, the examples are similar to Problems #1 and #2, but now students begin with equations. It is important for them to become flexible in translating between the different representations.

  In each example, students deal with two relationships at once. This makes it easier to compare the graphs, because they are in the same coordinate system. Students may need suggestions for showing all of the numbers in the table portion of the template. I have shown one way to do this in the Solutions, but there may be other possibilities. Students may also make tables on their own paper if they prefer.

- Problem #5

  In these examples, students deal for the first time with the effects of using different scales on the *x*- and *y*-axes. Encourage them to pay close attention to the rates of change. It is no longer a matter of merely counting squares on the grid.

  In the second example, students may find themselves showing fractions or decimals in their tables for the first time.

  If they have not before now, students may begin to discover that they can use the rate of change and the starting number to help them find their equations. The format is

  $$y = (\text{rate of change}) \cdot x + (\text{starting number})$$

  Be sure to allow students to write their equations in the format that they choose. For example, some of them may write the starting number first. This may make more sense to them, because they can imagine beginning with that number and adding multiples of the rate of change to it.

  Algebra texts show this format as $y = mx + b$. By allowing students to create their own strategies before encountering this formula in a beginning algebra course, students begin to develop important intuitions that will later help them understand why the formula *makes sense*!

- Problem #6

  In these examples, students do not see the formula, table, or graph (except in the second example where they see a single point on the graph). Instead, they begin with information about the starting number and rate of change.

  Also, students need to set up their own scales on the graphs in this problem. Since they may choose to do this in different ways, your students' creative approaches may not always match what you see in the Solutions.

- Problem #7

  These problems get students involved with "messier" rates of change. The first example may be especially challenging, because students need to divide a difference of 3 into 7 equal parts. Many will choose to use decimals. It is a great thinking exercise to ask them to write the number in fraction form as well. In my experience, though they usually understand that they can write $\frac{3}{7}$ in decimal form by dividing 3 ÷ 7, they are not nearly as quick to see that 3 ÷ 7 is *equal* to $\frac{3}{7}$! (This may result from stressing *procedures* over *concepts*.)

Coming up with stories for numbers like these, especially in the first example, can be very challenging. (As I had mentioned, my story is quite contrived!) If students are having trouble, they may choose to write stories that are imaginative rather than realistic. The most important thing is that the stories reflect a correct mathematical understanding of the concepts.

**Problem # 1**

*linear?*

*rate of change:*

*starting number:*

*formula:*

*story:*

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 1 | 2 |
| 2 | 4 |
| 3 | 6 |
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*linear?*

*rate of change:*

*starting number:*

*formula:*

*story:*

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 2 | 1 |
| 4 | 2 |
| 6 | 3 |
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**Sample Solutions for #1**

*linear?* yes

*rate of change:* 2

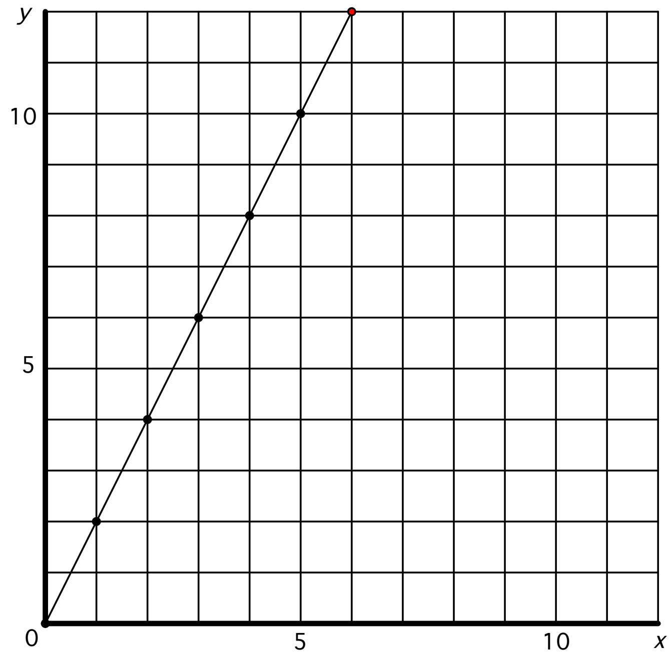
*starting number:* 0

*formula:* $y = 2x$

*story:*

Aditya is buying small bags of treats for his dog. Each bag costs $2.00.
x: number of bags
y: cost

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 1 | 2 |
| 2 | 4 |
| 3 | 6 |
| 4 | 8 |
| 5 | 10 |
| 6 | 12 |
| 7 | 14 |
| 8 | 16 |
| 9 | 18 |
| 10 | 20 |




*linear?* yes

*rate of change:* 1/2 or 0.5

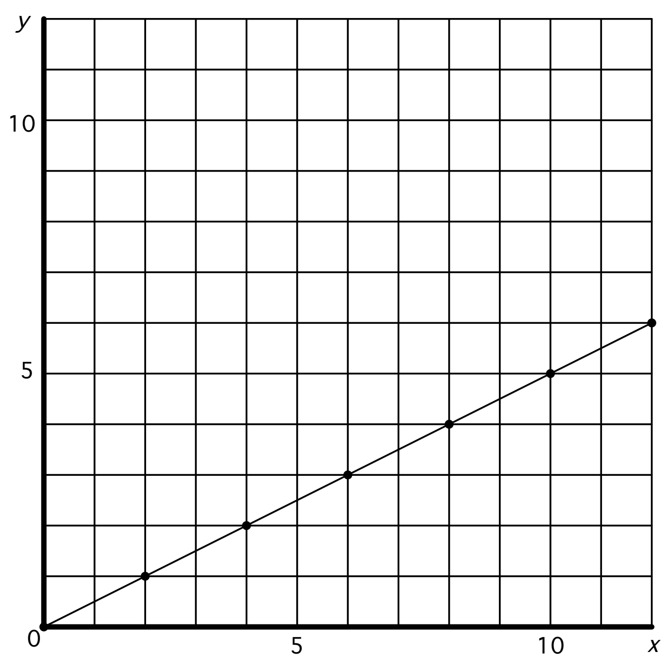
*starting number:* 0

*formula:* $y = 0.5x$

*story:*

Aditya is buying small bags of treats for his dog. Each bag costs $2.00.
x: cost
y: number of bags

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 2 | 1 |
| 4 | 2 |
| 6 | 3 |
| 8 | 4 |
| 10 | 5 |
| 12 | 6 |
| 14 | 7 |
| 16 | 8 |
| 18 | 9 |
| 20 | 10 |



**Problem #2**

linear?

rate of change:

starting number:

formula:

story:

| $x$ | $y$ |
|---|---|
| 0 | -1 |
| 1 | 1 |
| 2 | 3 |
| 3 | 5 |
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linear?

rate of change:

starting number:

formula:

story:

| $x$ | $y$ |
|---|---|
| 0 | 3 |
| 1 | 5 |
| 2 | 7 |
| 3 | 9 |
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**Sample Solutions for #2**

*linear?* yes

*rate of change:* 2

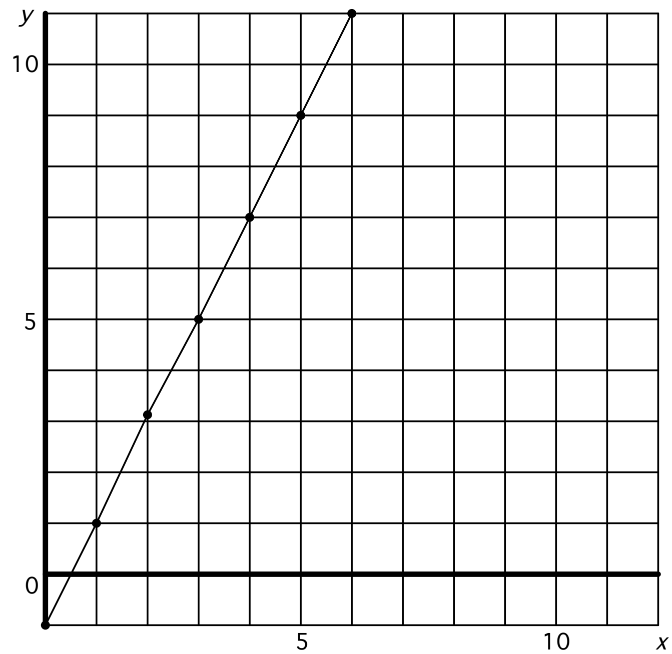
*starting number:* -1

*formula:* $y = 2x - 1$

*story:*

The temperature starts at -1°F and climbs 2°F each hour.
x: number of hours
y: temperature (°F)

| $x$ | $y$ |
|---|---|
| 0 | -1 |
| 1 | 1 |
| 2 | 3 |
| 3 | 5 |
| 4 | 7 |
| 5 | 9 |
| 6 | 11 |
| 7 | 13 |
| 8 | 15 |
| 9 | 17 |
| 10 | 19 |




*linear?* yes

*rate of change:* 2

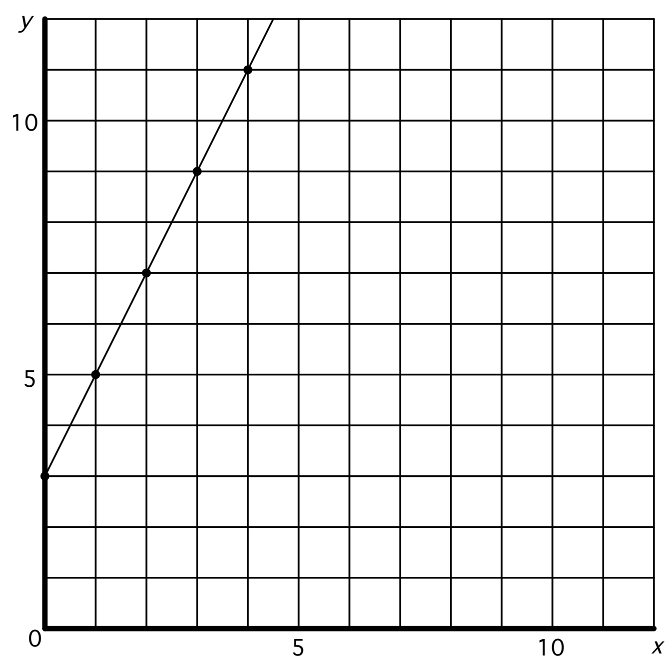
*starting number:* 3

*formula:* $y = 2x + 3$

*story:*

The temperature starts at 3°F and climbs 2°F each hour.
x: number of hours
y: temperature (°F)

| $x$ | $y$ |
|---|---|
| 0 | 3 |
| 1 | 5 |
| 2 | 7 |
| 3 | 9 |
| 4 | 11 |
| 5 | 13 |
| 6 | 15 |
| 7 | 17 |
| 8 | 19 |
| 9 | 21 |
| 10 | 23 |



# Problem #3

*linear?*

*rate of change:*

*starting number:*

*formula:*

*story:*

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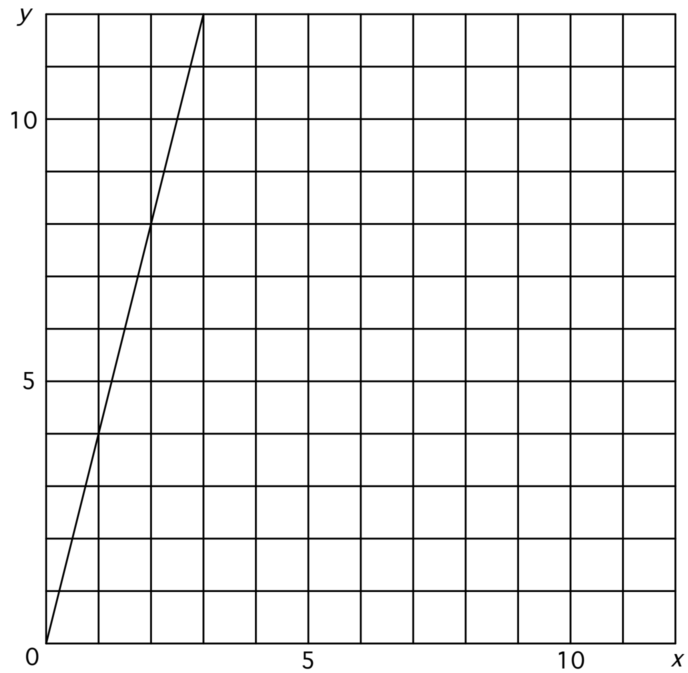


*linear?*

*rate of change:*

*starting number:*

*formula:*

*story:*

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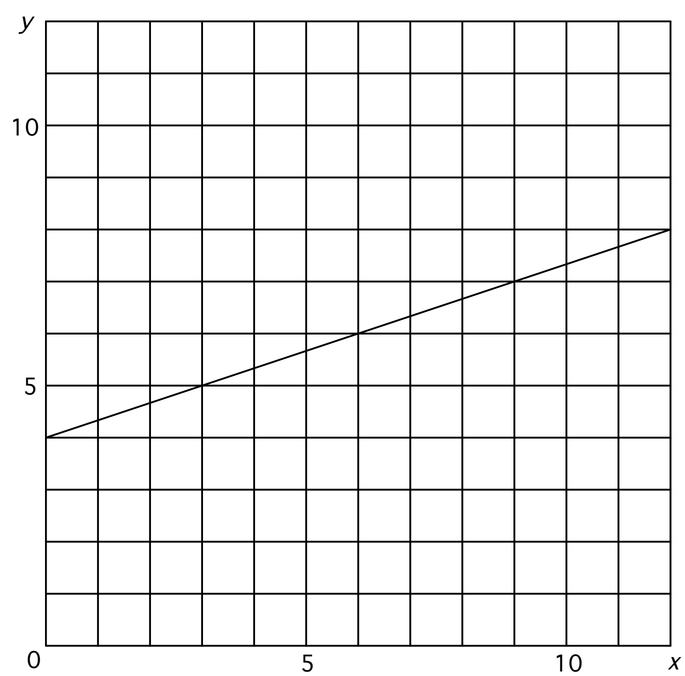


**Sample Solutions for #3**

*linear?* yes

*rate of change:* 4

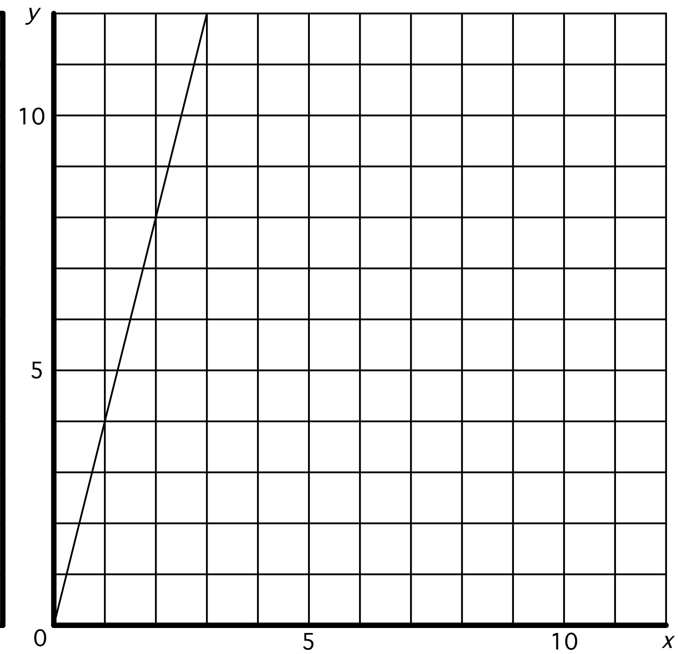
*starting number:* 0

*formula:* $y = 4x$

*story:*

Every button in a collection has 4 holes.
x: number of buttons
y: number of holes

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 1 | 4 |
| 2 | 8 |
| 3 | 12 |
| 4 | 16 |
| 5 | 20 |
| 6 | 24 |
| 7 | 28 |
| 8 | 32 |
| 9 | 36 |
| 10 | 40 |




*linear?* yes

*rate of change:* 1/3

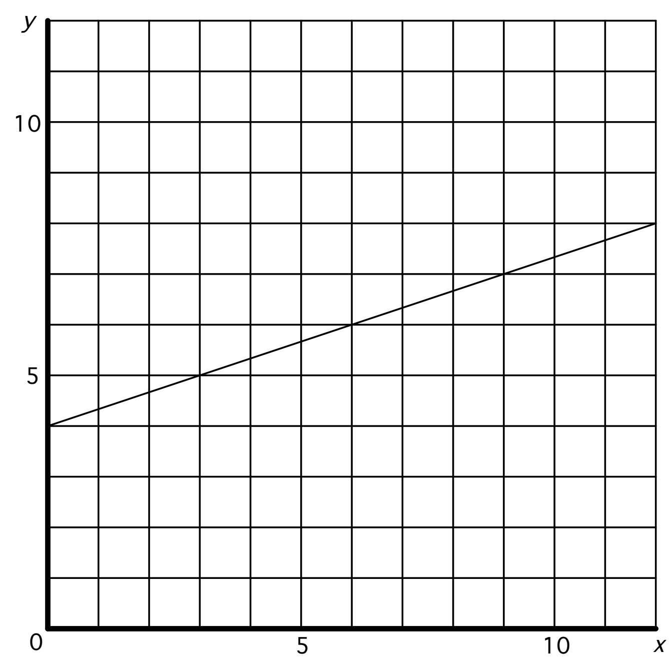
*starting number:* 4

*formula:* $y = (1/3)x + 4$

*story:*

A bucket has 4 cups of water in it. 1/3 of a cup of water drips into it every hour.
x: number of hours
y: number of cups

| $x$ | $y$ |
|---|---|
| 0 | 4 |
| 1 | $4\frac{1}{3}$ |
| 2 | $4\frac{2}{3}$ |
| 3 | 5 |
| 4 | $5\frac{1}{3}$ |
| 5 | $5\frac{2}{3}$ |
| 6 | 6 |
| 7 | $6\frac{1}{3}$ |
| 8 | $6\frac{2}{3}$ |
| 9 | 7 |
| 10 | $7\frac{1}{3}$ |



## Problem #4

*linear?*

*rate of change:*

*starting number:*

*formula:* $y = 2x + 1$ $\quad y = 2x - 3$

*story:*

*linear?*

*rate of change:*

*starting number:*

*formula:* $y = 0.5x - 1$ $\quad y = 0.5x + 2$

*story:*

## Sample Solutions for #4

*linear?* yes

*rate of change:* 2 , 2

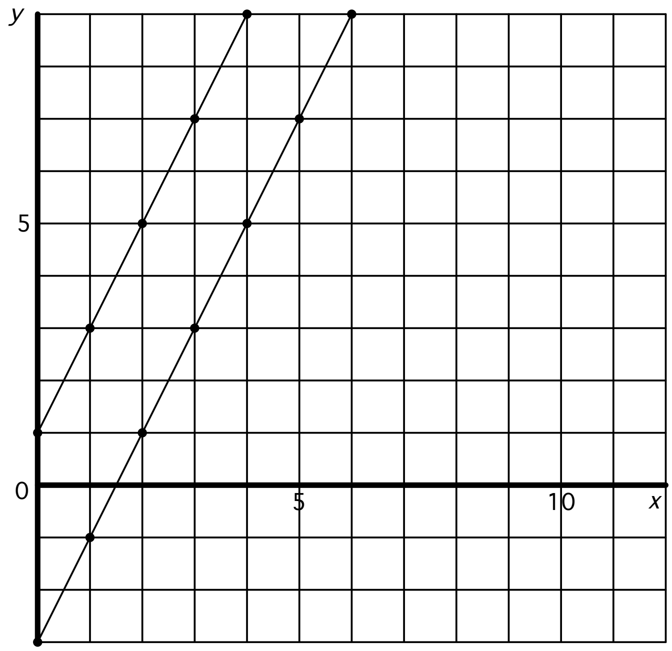
*starting number:* 1, -3

*formula:* $y = 2x + 1$ $y = 2x - 3$

*story:*

One frog starts 1 foot beyond the starting line and jumps 2 feet forward each time. Another frog starts 3 feet behind the starting line and jumps 2 feet forward each time.
x: number of jumps
y: number of feet beyond the starting line.

| x | y | x | y |
|---|---|---|---|
| 0 | 1 | 0 | -3 |
| 1 | 3 | 1 | -1 |
| 2 | 5 | 2 | 1 |
| 3 | 7 | 3 | 3 |
| 4 | 9 | 4 | 5 |
| 5 | 11 | 5 | 7 |
| 6 | 13 | 6 | 9 |
| 7 | 15 | 7 | 11 |
| 8 | 17 | 8 | 13 |
| 9 | 19 | 9 | 15 |
| 10 | 21 | 10 | 21 |




*linear?* yes

*rate of change:* 0.5 , 0.5

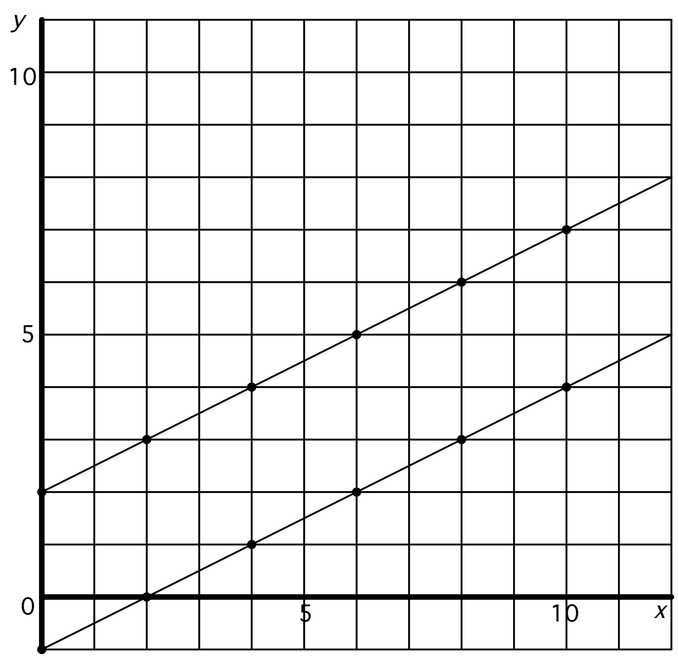
*starting number:* -1 , 2

*formula:* $y = 0.5x - 1$ $y = 0.5x + 2$

*story:*

One frog starts 1 foot behind beyond the starting line and jumps 0.5 feet forward each time. Another frog starts 2 feet beyond the starting line and jumps 0.5 feet forward each time.
x: number of jumps
y: number of feet beyond the starting line.

| x | y | x | y |
|---|---|---|---|
| 0 | -1 | 0 | 2 |
| 1 | -0.5 | 1 | 2.5 |
| 2 | 0 | 2 | 3 |
| 3 | 0.5 | 3 | 3.5 |
| 4 | 1 | 4 | 4 |
| 5 | 1.5 | 5 | 4.5 |
| 6 | 2 | 6 | 5 |
| 7 | 2.5 | 7 | 5.5 |
| 8 | 3 | 8 | 6 |
| 9 | 3.5 | 9 | 6.5 |
| 10 | 4 | 10 | 7 |



**Problem #5**

*linear?*

*rate of change:*

*starting number:*

*formula:*

*story:*

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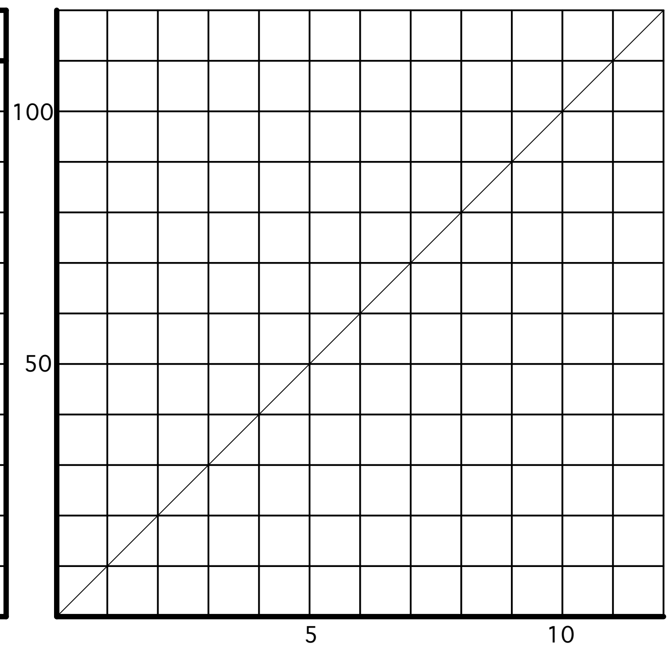


*linear?*

*rate of change:*

*starting number:*

*formula:*

*story:*

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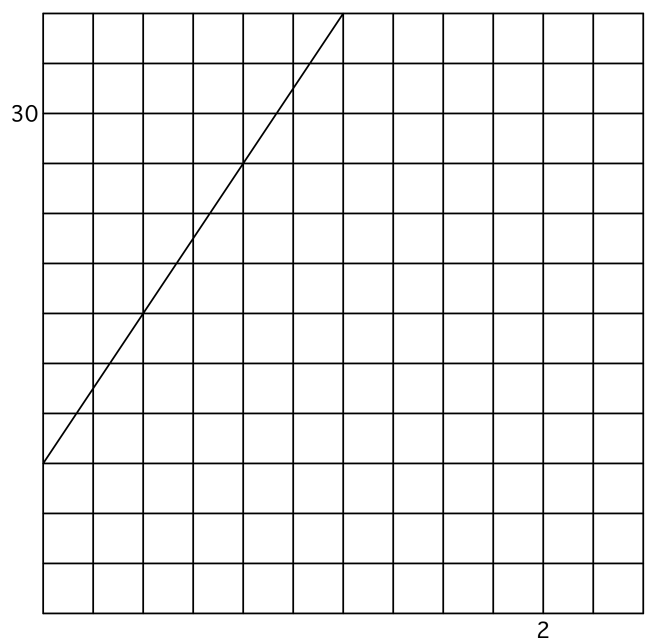


**Sample Solutions for #5**

*linear?* yes

*rate of change:* 10

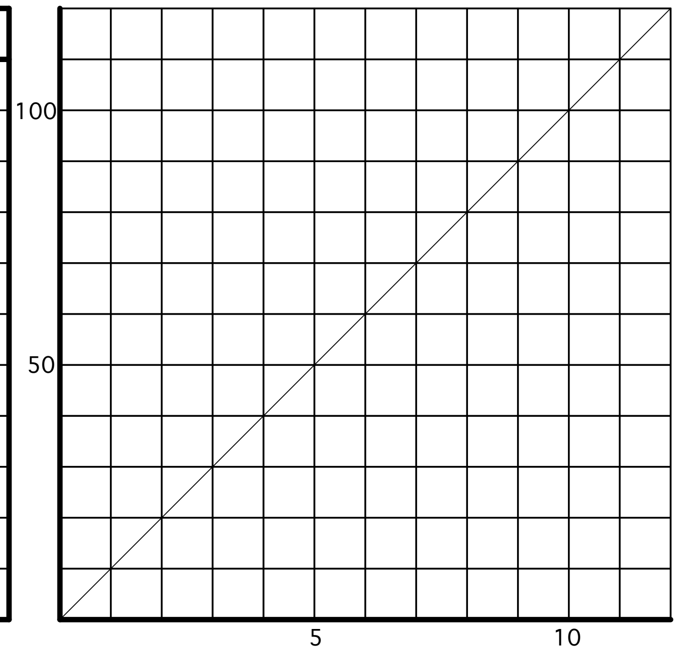
*starting number:* 0

*formula:* $y = 10x$

*story:*

Kylie earns 10 dollars per hour working at a clothing store.
x: number of hours worked
y: earnings in dollars

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 1 | 10 |
| 2 | 20 |
| 3 | 30 |
| 4 | 40 |
| 5 | 50 |
| 6 | 60 |
| 7 | 70 |
| 8 | 80 |
| 9 | 90 |
| 10 | 100 |




*linear?* yes

*rate of change:* 22.5

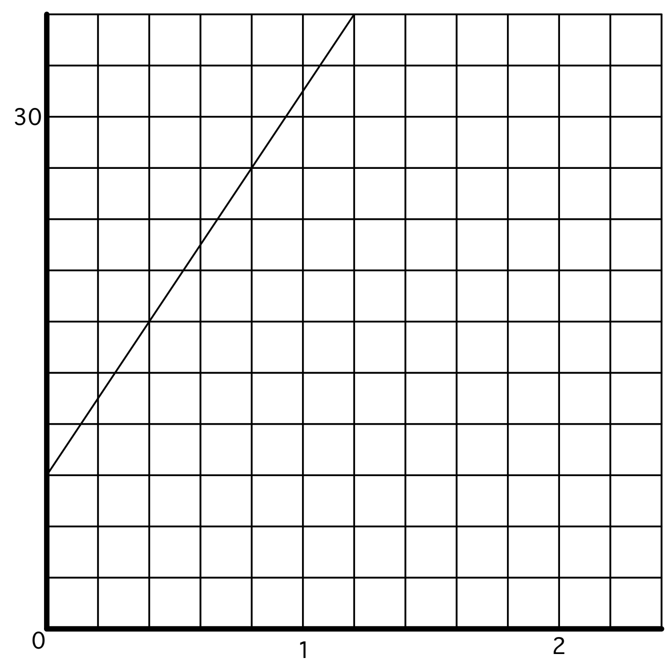
*starting number:* 9

*formula:* $y = 22.5x + 9$

*story:*

An icicle weighs 9 ounces and steadily accumulates ice at a rate of 22.5 ounces each day.
x: number of days
y: number of ounces

| $x$ | $y$ |
|---|---|
| 0 | 9 |
| 0.2 | 13.5 |
| 0.4 | 18 |
| 0.6 | 22.5 |
| 0.8 | 27 |
| 1 | 31.5 |
| 1.2 | 36 |
| 1.4 | 40.5 |
| 1.6 | 45 |
| 1.8 | 49.5 |
| 2 | 54 |



## Problem #6

*linear?* yes

*rate of change:* 6

*starting number:* -2

*formula:*

*story:*

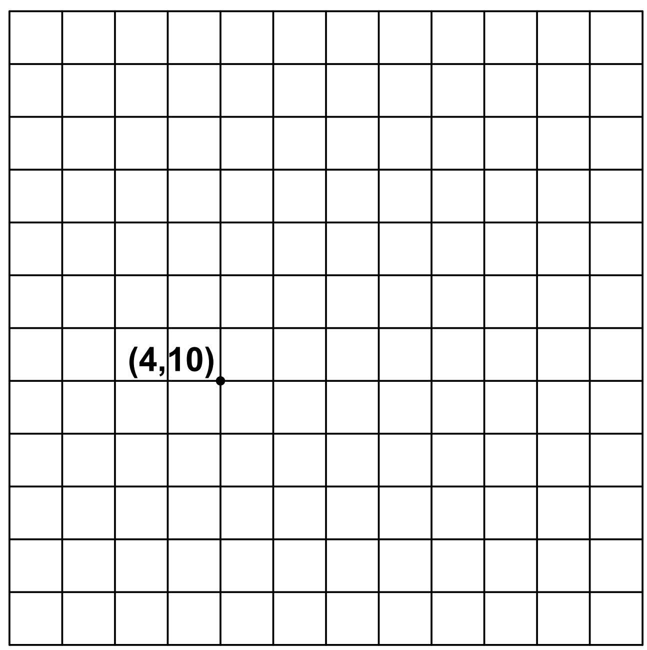
*linear?* yes

*rate of change:* 0.75

*starting number:*

*formula:*

*story:*



**Sample Solutions for #6**

*linear?* yes

*rate of change:* 6

*starting number:* -2

*formula:* $y = 6x - 2$

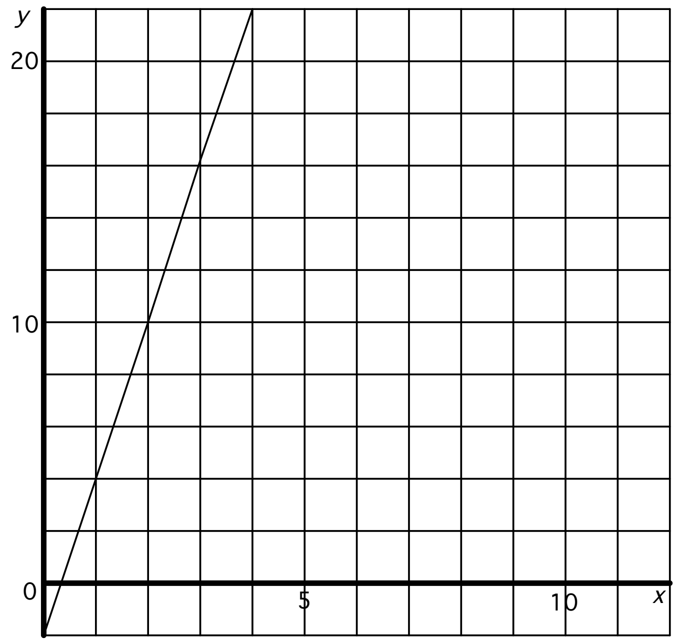
*story:*

Jack currently has -2 points in a card game. He earns 6 points for every card he plays in his next turn.

x: cards played

y: new score

| $x$ | $y$ |
|---|---|
| 0 | -2 |
| 1 | 4 |
| 2 | 10 |
| 3 | 16 |
| 4 | 22 |
| 5 | 28 |
| 6 | 34 |
| 7 | 40 |
| 8 | 46 |
| 9 | 52 |
| 10 | 58 |




*linear?* yes

*rate of change:* 0.75

*starting number:* 7

*formula:* $y = 0.75x + 7$

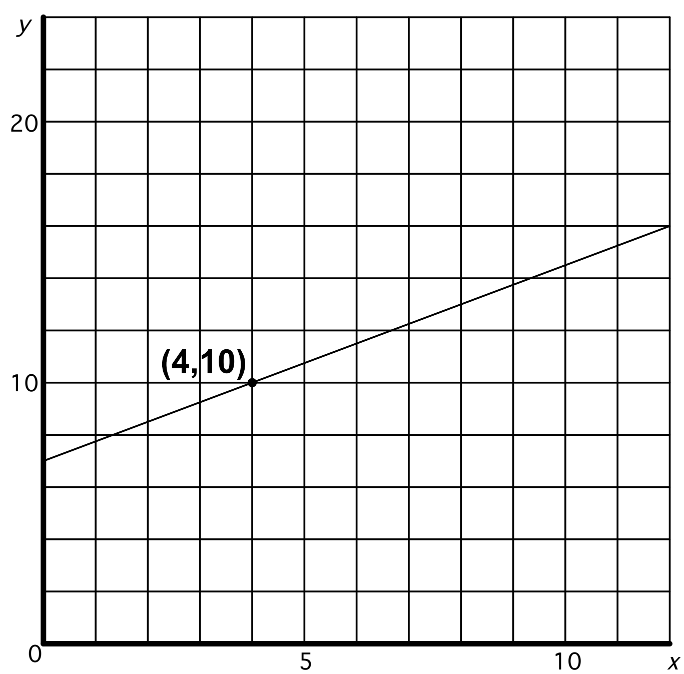
*story:*

The value of a classic car starts at about $7000 and increases by about $750 per year.

x: number of years

y: value of car in thousands of dollars

| $x$ | $y$ |
|---|---|
| 0 | 7 |
| 1 | 7.75 |
| 2 | 8.5 |
| 3 | 9.25 |
| 4 | 10 |
| 5 | 10.75 |
| 6 | 11.5 |
| 7 | 12.25 |
| 8 | 13 |
| 9 | 13.75 |
| 10 | 14.5 |



**Problem #7**

*linear?* yes

*rate of change:*

*starting number:*

*formula:*

*story:*

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| 2 | 3 |
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| | |
| 9 | 6 |
| | |

*linear?* yes

*rate of change:*

*starting number:*

*formula:*

*story:*

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| 60 | 50 |
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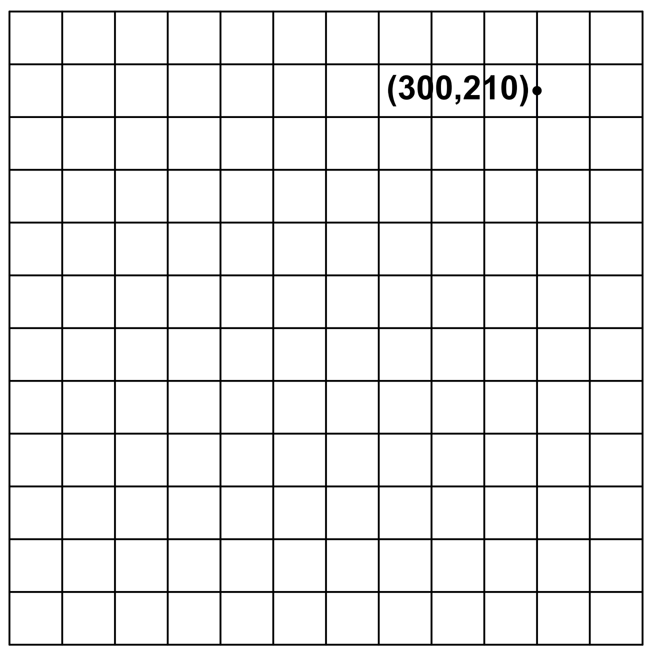


## Sample Solutions for #7

*linear?* yes

*rate of change:* $\frac{3}{7}$

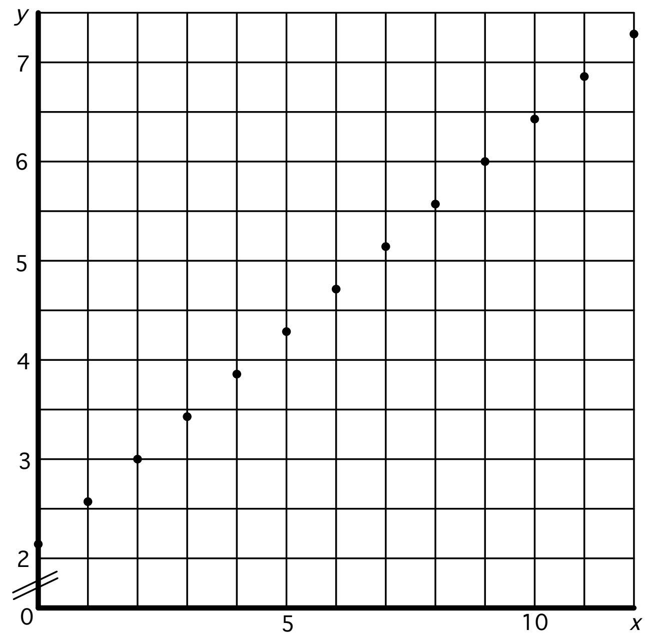
*starting number:* $2\frac{1}{7}$

*formula:* $y = \frac{3}{7}x + 2\frac{1}{7}$

*story:*

Ginny's birthday is on the 15th of the month. She cleans her room once every 3 days.
x: number of times Ginny has cleaned her room since her birthday
y: number of weeks since the beginning of Ginny's birth month

| $x$ | $y$ |
|---|---|
| 0 | $2\frac{1}{7}$ |
| 1 | $2\frac{4}{7}$ |
| 2 | 3 |
| 3 | $3\frac{3}{7}$ |
| 4 | $3\frac{6}{7}$ |
| 5 | $4\frac{2}{7}$ |
| 6 | $4\frac{5}{7}$ |
| 7 | $5\frac{1}{7}$ |
| 8 | $5\frac{4}{7}$ |
| 9 | 6 |
| 10 | $6\frac{3}{7}$ |




*linear?* yes

*rate of change:* $\frac{2}{3}$

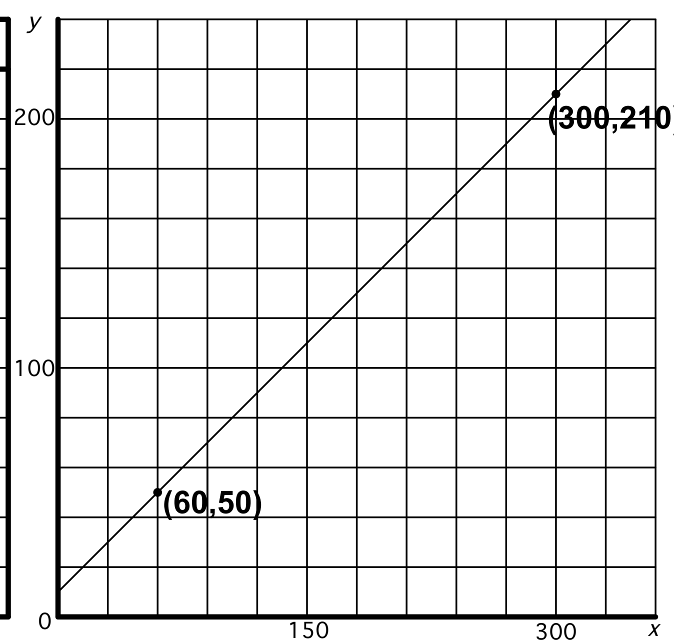
*starting number:* 10

*formula:* $y = \frac{2}{3}x + 10$

*story:*

Sam read 10 books in January. Since then he has read two books every three days.
x: number days since the end of January
y: number of books Sam has read since the beginning of the year.

| $x$ | $y$ |
|---|---|
| 60 | 50 |
| 90 | 70 |
| 120 | 90 |
| 150 | 110 |
| 180 | 130 |
| 210 | 150 |
| 240 | 170 |
| 270 | 190 |
| 300 | 210 |
| 330 | 230 |
| 360 | 250 |



## Stage 2

In the Stage 2 problems, students explore negative rates of change, linear relationships whose graphs intersect, and relationships whose numbers are especially "messy" (something that happens often when applying these ideas in real-world situations).

**What students should know**

- Understand concepts from Stage 1.

**What students will learn**

- Understand and represent negative rates of change from the perspectives of numbers, graphs, equations, and real-world contexts.
- Explore intersections on graphs of linear relationships from the perspective of numbers, graphs, equations, and real-world contexts.
- Analyze and apply concepts of linear relationships in situations involving "messy" and/or approximate numbers.

**Notes on specific problems**

- Problem #8
  Students begin exploring negative rates of changes by dealing with a situation in which the outputs in a table decrease while the inputs increase. They learn to recognize that the graph of a relation with a negative rate of change drops from left to right.

  They also learn that previously discovered patterns continue to hold when rates of change are negative. For example, equations for these relationships still fit the general form

  output = (rate of change) · input + (starting number)

  or $y = mx + b$, and the starting number still appears as the point where a graph crosses the $y$-axis.

- Problem #9
  These examples are similar to those in Problem #8, but students begin with different given information—an equation in the top example and a graph in the bottom one.

In the top example, all of the outputs will be negative (if students choose positive inputs), and students need to set up their own scales on the axes. In the bottom example, the scales are shown, but students need to think carefully in order to determine the rate of change.

- Problem #10
  Each example shows a pair of intersecting linear graphs. (Refer to Problem #4 for ideas about displaying the information in the tables on the template.) In addition to finding the usual information, students may become curious about where the graphs intersect. (If they do not, you should draw their attention to this!) They develop their own strategies for finding the coordinates of these points. It makes sense to begin with an estimate. Some students may be able to find more precise values algebraically, but many will probably use trial and error combined with proportional reasoning, especially in the bottom example where the coordinates are not whole numbers.

  Ask students to discuss the significance of the points of intersection as they relate to the stories that they have created. And as always, encourage them to share and compare their stories and strategies!

- Problem #11
  The concepts in Problem #11 are similar to those in earlier problems, but the messy numbers will probably make the experience feel quite different. If your students struggle at first (which is likely), ask them to refer to the strategies and concepts from earlier problems and to adapt them to these new situations. Keeping foundational ideas in sight when faced with the complexity of messy calculations can be an important part of consolidating and extending the learning of new concepts. This is especially true in the case of linear relationships, because real-world applications so often involve messy and/or approximate values. Note: The bottom example poses the additional challenge of showing negative inputs in the graph.

  Creating realistic stories may be an especially challenging task for these examples. I have left this particular challenge for you and your students! I hope that you will share some of your stories with me at
  jburkhart@5280math.com/contact.
  With your permission, I may even include some of them in the Solutions (giving you credit, of course)!

**Problem #8**

*linear?*

*rate of change:*

*starting number:*

*formula:*

*story:*

| | |
|---|---|
| 0 | 7 |
| 1 | 6 |
| 2 | 5 |
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |

*linear?* yes

*rate of change:*

*starting number:*

*formula:*

*story:*

| | |
|---|---|
| 0 | 15 |
| | |
| 2 | 10 |
| | |
| | |
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## Sample Solutions for #8

*linear?* yes

*rate of change:* -1

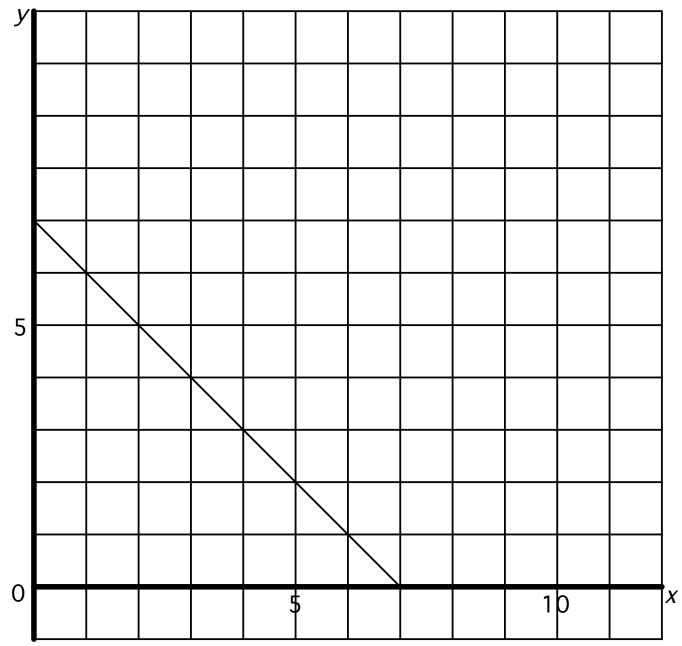
*starting number:* 7

*formula:* $y = -x + 7$ or $y = 7 - x$

*story:*

Nikita has $7. Each day for the next week, she spends $1 to buy a snack.
x: number of days
y: amount of money Nikita has left

| $x$ | $y$ |
|---|---|
| 0 | 7 |
| 1 | 6 |
| 2 | 5 |
| 3 | 4 |
| 4 | 3 |
| 5 | 2 |
| 6 | 1 |
| 7 | 0 |
| | |
| | |
| | |




*linear?* yes

*rate of change:* -2.5

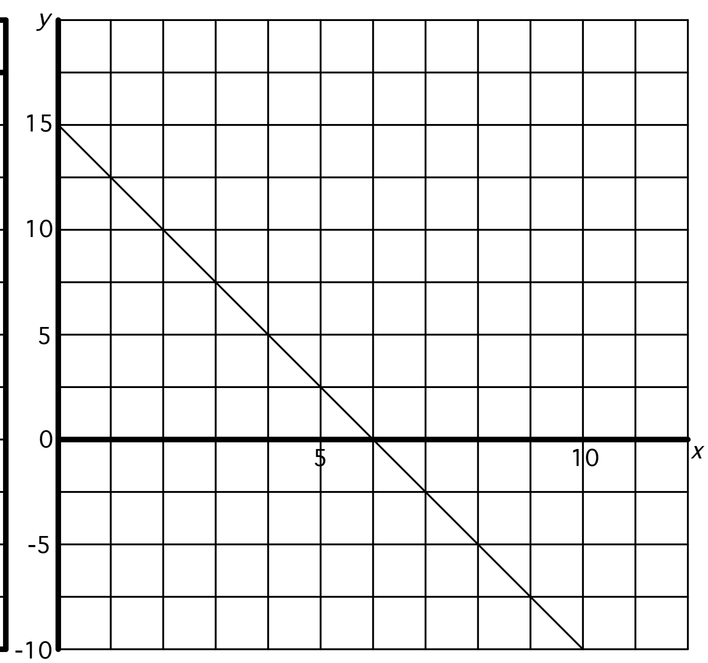
*starting number:* 15

*formula:* $y = -2.5x + 15$ or $y = 15 - 2.5x$

*story:*

Jessamine's company has $15,000 in an account to cover utility expenses. Each month, the company is billed $2500 for expenses.
x: number of months
y: balance of the account (thousands of dollars)

| $x$ | $y$ |
|---|---|
| 0 | 15 |
| 1 | 12.5 |
| 2 | 10 |
| 3 | 7.5 |
| 4 | 5 |
| 5 | 2.5 |
| 6 | 0 |
| 7 | -2.5 |
| 8 | -5 |
| 9 | -7.5 |
| 10 | -10 |



## Problem #9

*linear?*

*rate of change:*

*starting number:*

*formula:* $y = -0.25x - 5$

*story:*

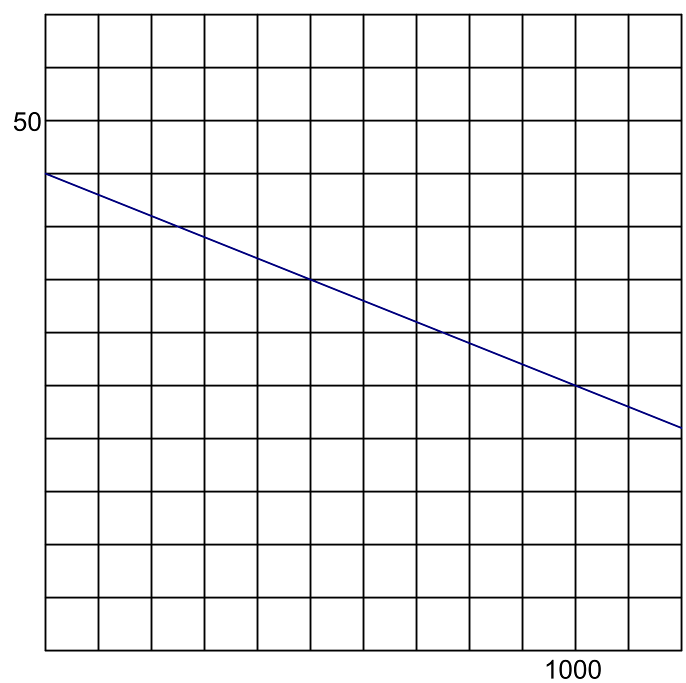
*linear?* yes

*rate of change:*

*starting number:*

*formula:*

*story:*



## Sample Solutions for #9

*linear?* yes

*rate of change:* -0.25

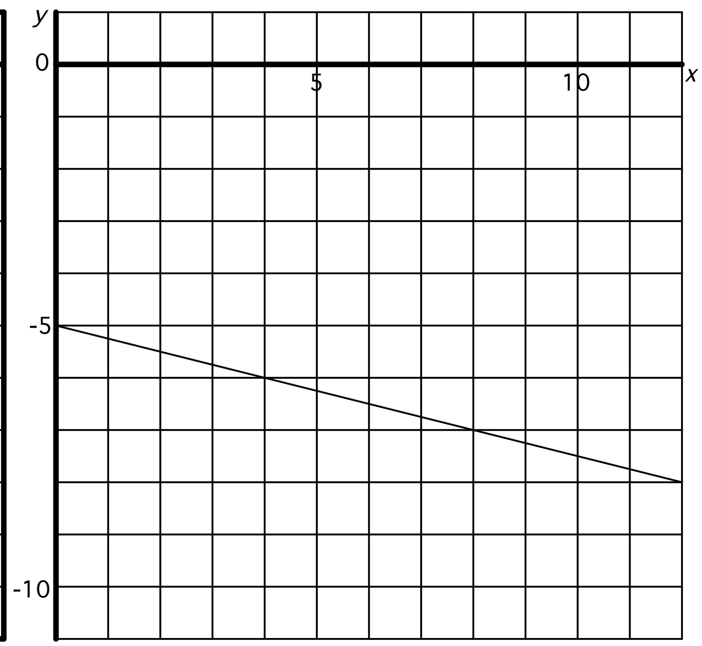
*starting number:* -5

*formula:* $y = -0.25x - 5$

*story:*

A bucket is being lowered slowly into a well.
x: time (seconds)
y: height of the bucket relative to the surface (meters)

| $x$ | $y$ |
|---|---|
| 0 | -5 |
| 1 | -5.25 |
| 2 | -5.5 |
| 3 | -5.75 |
| 4 | -6 |
| 5 | -6.25 |
| 6 | -6.5 |
| 7 | -6.75 |
| 8 | -7 |
| 9 | -7.25 |
| 10 | -7.5 |




*linear?* yes

*rate of change:* -0.02

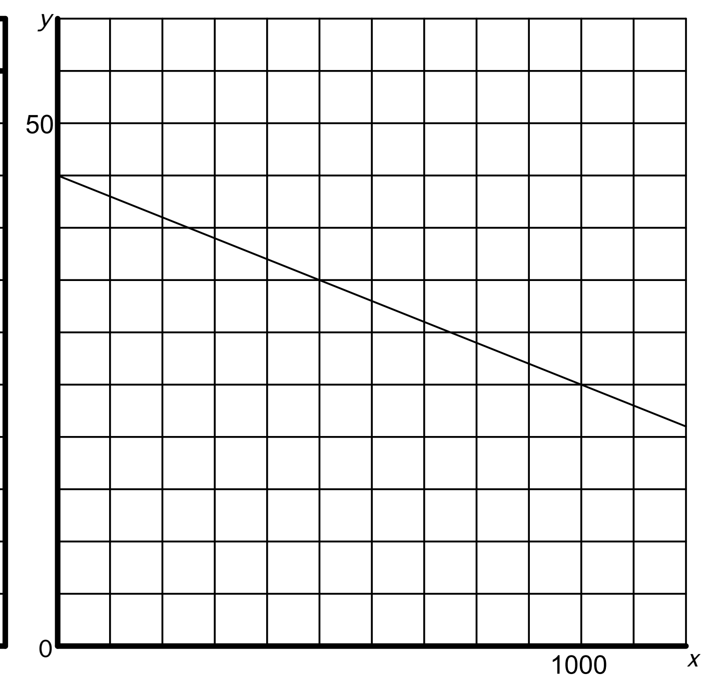
*starting number:* 45

*formula:* $y = -0.02x + 45$

*story:*

A glacier is melting in the summer.
x: number of hours
y: height of the front of the glacier in meters

| $x$ | $y$ |
|---|---|
| 0 | 45 |
| 100 | 43 |
| 200 | 41 |
| 300 | 39 |
| 400 | 37 |
| 500 | 35 |
| 600 | 33 |
| 700 | 31 |
| 800 | 29 |
| 900 | 27 |
| 1000 | 25 |



## Problem #10

*linear?*

*rate of change:*

*starting number:*

*formula:*

*story:*

| | |
|---|---|
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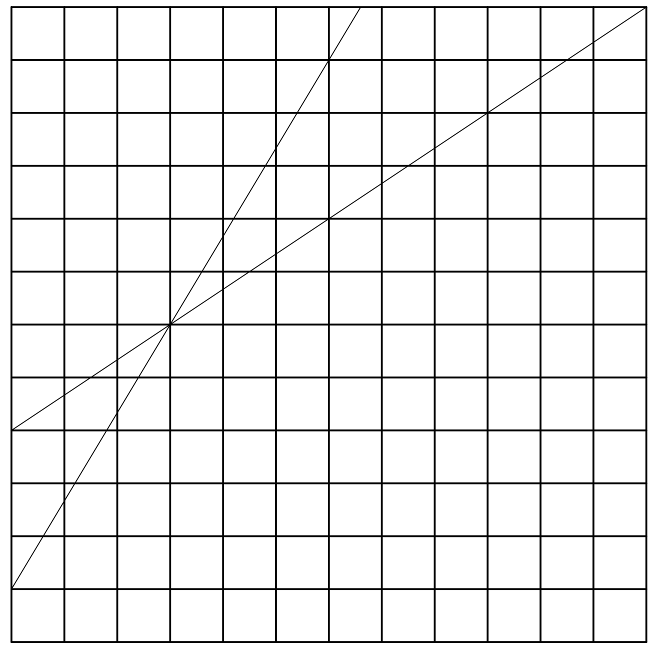

*linear?*

*rate of change:*

*starting number:*

*formula:*

*story:*

| | |
|---|---|
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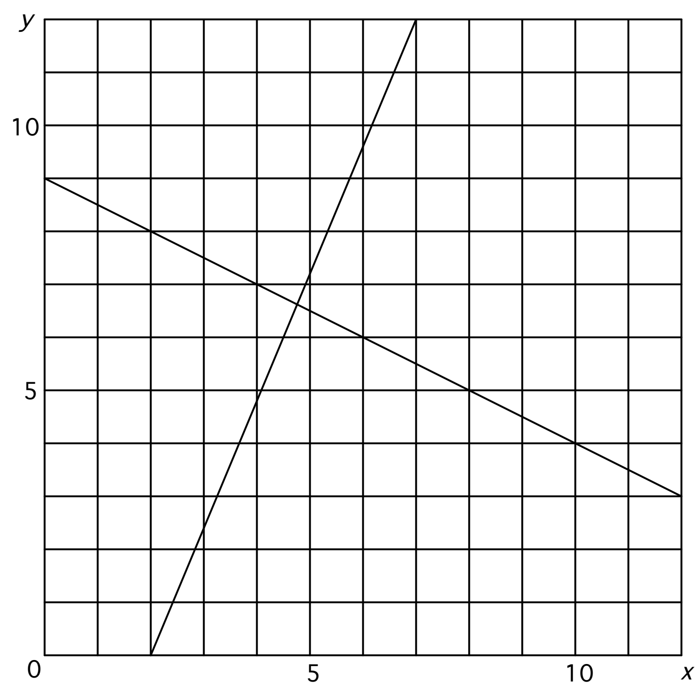


**Sample Solutions for #10**

*linear?* yes , yes

*rate of change:* 5 , 2

*starting number:* 2 , 12

*formula:* $y = 5x + 3$ $y = 2x + 12$

*story:*

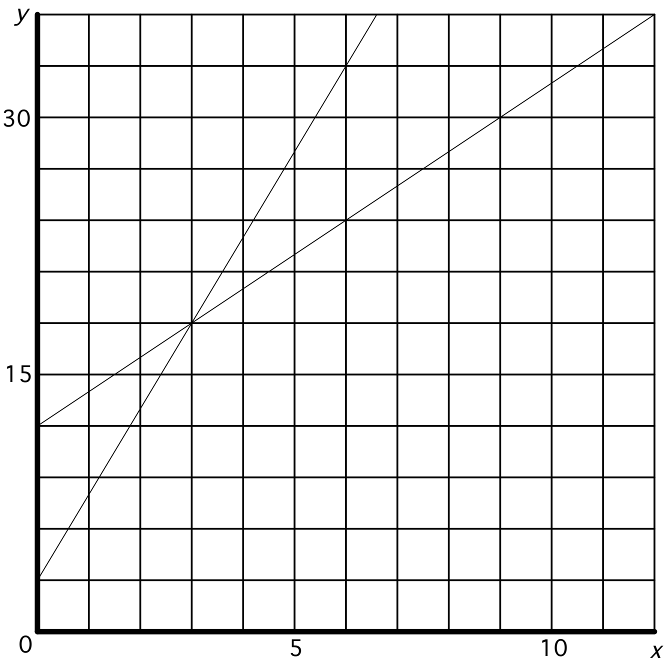
Ahmed has 3 dollars, and earns 5 dollars each week for mowing the lawn. Leticia has 12 dollars and earns 2 dollars each week.

x: number of weeks

y: amount of money they have (dollars)

Intersection point: approx (3 , 18)

| x | y | x | y |
|---|---|---|---|
| 0 | 3 | 0 | 12 |
| 1 | 8 | 1 | 14 |
| 2 | 13 | 2 | 16 |
| 3 | 18 | 3 | 18 |
| 4 | 23 | 4 | 20 |
| 5 | 28 | 5 | 22 |
| 6 | 33 | 6 | 24 |
| 7 | 38 | 7 | 26 |
| 8 | 43 | 8 | 28 |
| 9 | 48 | 9 | 30 |
| 10 | 53 | 10 | 32 |




*linear?* yes , yes

*rate of change:* 24 , -5

*starting number:* -48 , 90

*formula:* $y = 24x - 48$ $y = -5x + 90$

*story:*

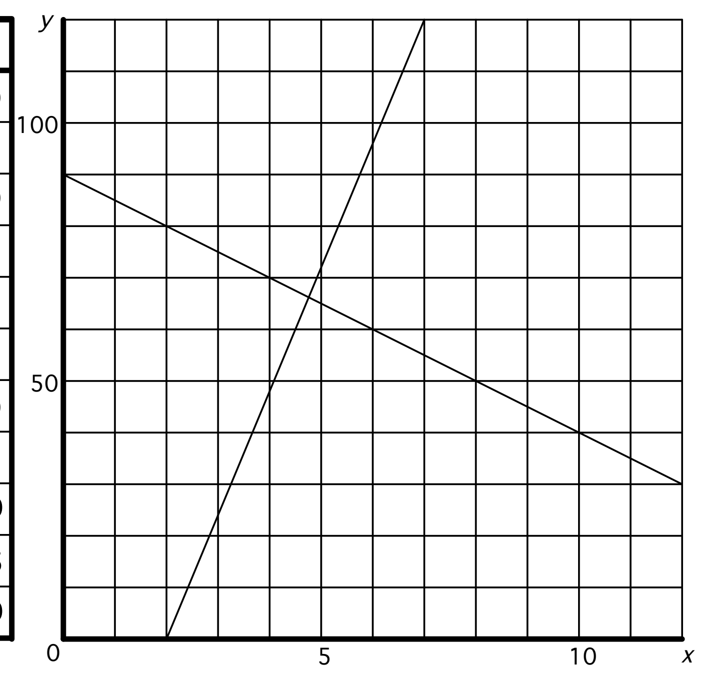
x: price that a company sets for an item it is selling

y (left side): number of items the company is able/willing to produce at that price

y (right): number of items that sell at that price

Intersection point: approx (4.76 , 66.2)

| x | y | x | y |
|---|---|---|---|
| | | 0 | 90 |
| | | 1 | 85 |
| 2 | 0 | 2 | 80 |
| 3 | 24 | 3 | 75 |
| 4 | 48 | 4 | 70 |
| 5 | 70 | 5 | 65 |
| 6 | 94 | 6 | 60 |
| 7 | 120 | 7 | 55 |
| | | 8 | 50 |
| | | 9 | 45 |
| | | 10 | 40 |



## Problem #11

*linear?* yes

*rate of change:*

*starting number:*

*formula:*

*story:*

| | |
|---|---|
| | |
| 300 | 172 |
| | |
| | |
| | |
| | |
| | |
| | |
| 580 | 997 |
| | |

*linear?*

*rate of change:*

*starting number:*

*formula:*

*story:*

| | |
|---|---|
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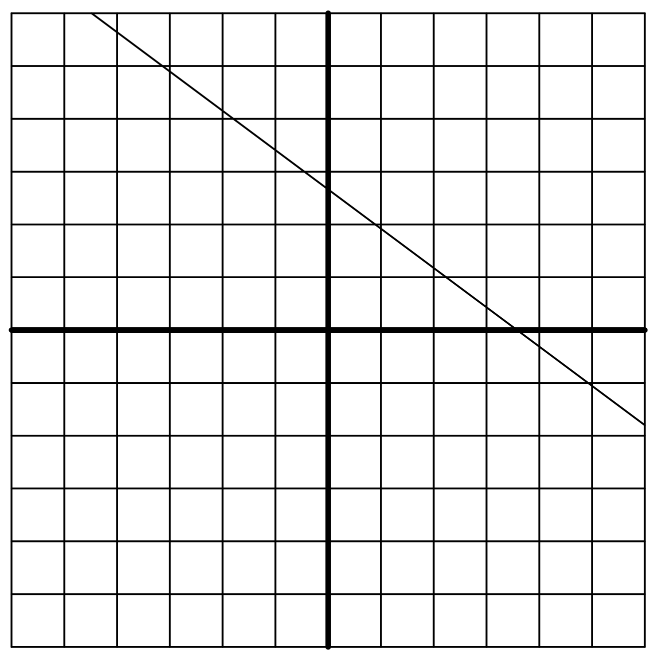


**Sample Solutions for #11**

*linear?* yes

*rate of change:* $\frac{165}{56}$ or $\approx 2.95$

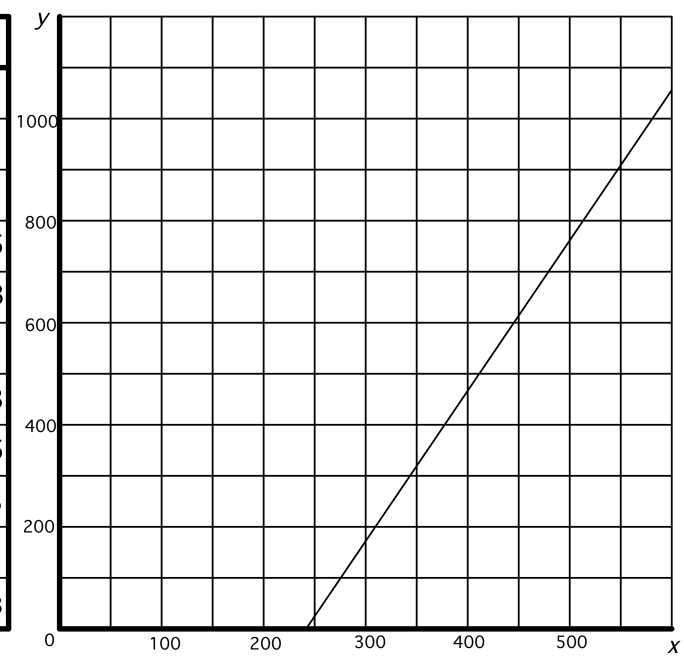
*starting number:* $-\frac{9967}{14} \approx -711.9$

*formula:* $y = \frac{165}{56}x - \frac{9967}{14}$ or $y \approx 2.95x - 711.9$

*story:*

answers will vary

| *x* | *y* |
|---|---|
| 265 | 68.88 |
| 300 | 172 |
| 335 | 275.13 |
| 370 | 378.25 |
| 405 | 481.38 |
| 440 | 584.5 |
| 475 | 687.63 |
| 510 | 790.75 |
| 545 | 893.88 |
| 580 | 997 |
| 615 | 1100.13 |




*linear?* yes

*rate of change:* about -0.75

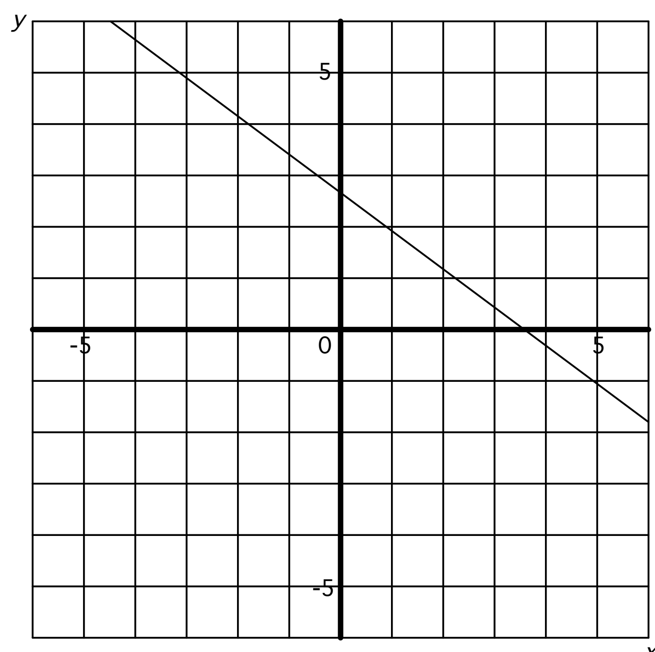
*starting number:* about 2.7

*formula:* $y \approx -0.75x + 2.7$

*story:*

answers will vary

| *x* | *y* |
|---|---|
| -4 | 5.7 |
| -3 | 4.95 |
| -2 | 4.2 |
| -1 | 3.45 |
| 0 | 2.7 |
| 1 | 1.95 |
| 2 | 1.2 |
| 3 | 0.45 |
| 4 | -0.3 |
| 5 | -1.05 |
| 6 | -1.8 |



## Stage 3

The Stage 3 problems offer a few new challenges for students: *non-linear* relationships, *inverse* relationships, and *asymptotes* (loosely speaking, lines that are approached ever more closely but not reached by a graph). Additionally, each example is related to a well-known (though not necessarily by students) real-world or mathematical context.

**What students should know**

- Understand concepts from Stages 1 and 2.

**What students will learn**

- Understand that non-linear relationships do not have constant rates of change and that their graphs are not lines.
- Find formulas for non-linear relationships.
- Enhance understanding of well-known formulas by analyzing them from the perspective of formulas, tables, and graphs.
- Discover and make use of patterns in the connections between mathematical relationships and their inverses.
- Informally explore the concept of an *asymptote* in the context of formulas, tables, real-world contexts, and graphs.

**Notes on specific problems**

- Problem #12

  The top example shows three pairs of numbers whose inputs and outputs could be diameters and circumferences of circles. If students use the variable *d* for the input and *c* for the output, their formula will be some version of the familiar one for calculating circumference from diameter: $c = \pi d$.

  The bottom example shows a graph whose inputs and outputs could be radii and areas of circles. By using *r* for the input and *A* for the output, students will produce some version of the familiar formula for calculating the area of a circle from its radius: $A = \pi r^2$.

  The bottom example is the first case of a non-linear relationship in this exploration. Students should notice that since the graph is not a line, the relationship does *not* have a constant rate of change, and the formula does *not* fit the pattern output = (rate of change) · input + (starting number).

The important distinction in this case is that the input (the radius) is squared.

The outputs for both examples are approximate because π is an irrational number. Because of this, the outputs shown in the Solutions for the top example appear to deviate slightly from a constant rate of change.

- Problem #13
  The first example shows three pairs of numbers whose inputs and outputs could be the number of sides of a polygon and the sum of its interior angles, respectively.

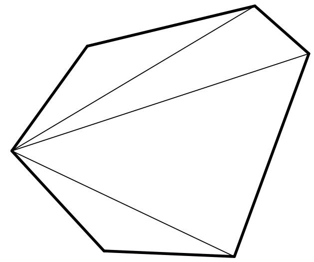
  While it would make numerical sense to fill in the top row in the table, it does not make geometric sense, since there are no 2-sided polygons. For this reason, I left it blank and did not include a point with an input of 2 on the graph. I also chose not to connect the dots on the graph because the number of sides is always a whole number. (Some of your students may choose to connect them in order to emphasize the linearity of the pattern.)

  The second formula in the Solutions, $a = 180n - 360$, is written in the familiar form

  output = (rate of change) · input + (starting number)

  for linear relationships. The starting number is negative in this case. Notice that the formula works in spite of the fact that the starting number does not make sense in the context of the geometry. That is, there are no 0-sided polygons (and there are certainly no polygons whose interior angles have a sum of –360°)! However, if you were to extend the pattern in the table upward, the output for 0 would be –360. And if you were to extend the graph to the left, it would cross the *y*-axis at –360.

  Some students may be more familiar with the first formula than the second one, because it comes from counting the number of triangles that one can draw from a single vertex. Notice that the number of triangles is two less than the number of sides, which explains the $n - 2$ in the formula.



Actually, you may understand both versions of the formula visually. For more information, see one of my *Creative Math Prompts* (5280math.com/nw-middle-5-hex-angles) or the activity, Polygon Perambulations, in the book *Advanced Common Core Explorations: Measurement and Polygons*.

The second example gives students an opportunity to explore *inverse* relationships. In an inverse relationship, the inputs and outputs are interchanged. Your students may notice a number of things:

- The columns in the tables are interchanged.
- Both relationships are linear.
- The rates of change are reciprocals of one another.
- If you were to extend the graphs, the *x*- and *y*-intercepts would exchange places. (The *x*-intercept is the place where the graph crosses the *x*-axis.)
- The equation for the inverse relationship entails doing the inverse operations in the opposite order. Specifically:

  In the top equation, you subtract 2, then multiply by 180.
  In the bottom equation, you divide by 180, then add 2.
- One equation is designed to calculate the angle sum from the number of sides, while the other calculates the number of sides from the angle sum.

Some students may also see a pattern between the graphs, though this is quite a bit harder to notice. The graphs are reflections of one another (across the line $y = x$).

Students will have more opportunities to observe patterns in inverse relationships in the remaining problems.

- Problem #14

  These two examples describe the relationship between the Fahrenheit and Celsius temperature scales—but try not to give this away. It is much more fun when the students discover it on their own. They should be impressed with their ability to derive these formulas for themselves based on knowing only the corresponding values at the freezing and boiling points of water!

In the second example, students again explore inverse relationships. They should make observations analogous to those in Problem #13. Specifically:

- The columns in the tables are interchanged.
- Both relationships are linear.
- The rates of change are reciprocals of one another ($\frac{9}{5}$ and $\frac{5}{9}$).
- If you were to extend the graphs, the *x*- and *y*-intercepts would exchange places.
- The equation for the inverse relationship entails doing the inverse operations in the opposite order. Specifically:

  In the top equation, you multiply by 1.8 ($\frac{9}{5}$), then add 32.

  In the bottom equation, you subtract 32, then divide by 1.8 (or, equivalently, multiply by its reciprocal, $\frac{5}{9}$).
- One equation is designed to calculate Fahrenheit from Celsius, while the other calculates Celsius from Fahrenheit.

As before, the graphs are symmetrical across the line $y = x$. Students may see this more easily if they draw both graphs in the same coordinate system. Keep in mind that this may feel confusing conceptually, because the units on the axes are different for each graph.

You may the extend the problem by posing more questions. For example:

(1) What (in terms of scales on thermometers) does it mean that the relationship between the two scales is linear? (When one scale changes by a fixed amount, the other changes by a corresponding fixed amount. This shows up in the observation that the degrees are equally spaced on both scales.)

(2) Is there a temperature at which both scales have the same numerical value? (Yes. –40°F and –40°C both represent the same temperature. Encourage students to use all three representations [tables, graphs, and formulas] to reach this conclusion.)

- Problem #15

  These examples show the approximate relationship between the interest rate (as a percentage) in a bank account and the amount of time it takes the account to double in value. Since students are unlikely to be familiar with this real-world context, you may want to let them try to invent their own contexts before sharing this one with them.

  Students may find it challenging to read many of the *x*- and *y*-coordinates from the graphs, but there are enough points that intersect at grid lines that, by beginning with these, they may be able to identify patterns that guide them towards a formula before estimating the remaining coordinates.

  This problem illustrates a situation in which an inverse relationship is the same as the original relationship! I have shown this in the Solutions by exchanging the columns in the two tables as usual. However, students may struggle for a while with the fact that the scales on the *x*- and *y*-axes are different. Because of this, they may choose ordered pairs that do not show this relationship as clearly as in previous problems.

  There are a few other issues that make this problem more challenging.

  - It is not possible to attain a value of 0 for either the input or the output. If you were to extend the graphs upward and to the right, you would see that the graph approaches the *y*-axis (as it rises) and the *x*-axis (as you move to the right). However, it never reaches either axis. Due to this behavior, the *x*- and *y*- axes are called *asymptotes* for the graph. Students will learn much more about asymptotes in advanced algebra and calculus courses.
  - The region of the graph shown in the problem is not necessarily the most useful part to look at from the real-world perspective of interest rates. As a further challenge, you might want to ask your students to draw regions of the graph that are more suitable to the real-world context.

  Again, students explore inverse relationships in this problem. They should make analogous observations. Specifically:

  - The columns in the tables are interchanged (though this may not be immediately apparent depending on the values they choose to include).
  - Neither relationship is linear.
  - The rates of change are not constant for either relationship.

- Although neither graph has an *x*- or a *y*-intercept, the graphs behave in the same fashion as they approach each axis.
- The act of dividing 70 by the input undoes itself. In other words, if you divide 70 by a number and then divide 70 by the result, you will get the original number back. This observation explains why the graphs and equations of the inverses are the same. This type of relationship is said to be "its own inverse." You may like to challenge your students to search for other examples of relationships that are their own inverses.
- One equation is designed to calculate the doubling time from the interest rate, while the other calculates the interest rate needed to achieve a chosen doubling time.

As before, the graphs are symmetrical across the line $y = x$. In this case, the graph itself is symmetric across this line! Consequently, reflecting it across this line does not change it.

Note: You may want to spend some time leading a discussion about what the asymptotes mean in the context of the back account.

- You will never be able to obtain a doubling time of 0 no matter how large the interest rate is, though you could theoretically make the time as short as you like by making the interest rate extremely high.
- By making the interest rate extremely small, the doubling time can be made as large as you like, but if the interest rate were 0%, the money would never double.

Students will learn more about doubling times when they study exponential functions in future algebra courses.

# Problem #12

*linear?*

*rate of change:*

*starting number:*

*formula:*

*story:*

| | |
|---|---|
| | |
| | |
| | |
| | |
| 2 | 6.28 |
| | |
| 3 | 9.42 |
| | |
| | |
| | |
| 5 | 15.71 |

*linear?*

*rate of change:*

*starting number:*

*formula:*

*story:*

| | |
|---|---|
| | |
| 1 | 3.14 |
| | |
| | |
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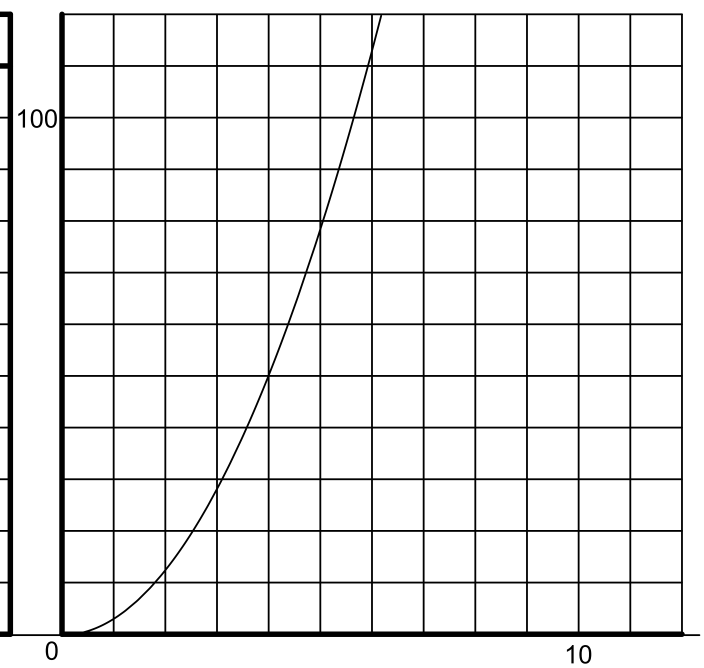


**Sample Solutions for #12**

*linear?* yes

*rate of change:* about 3.14

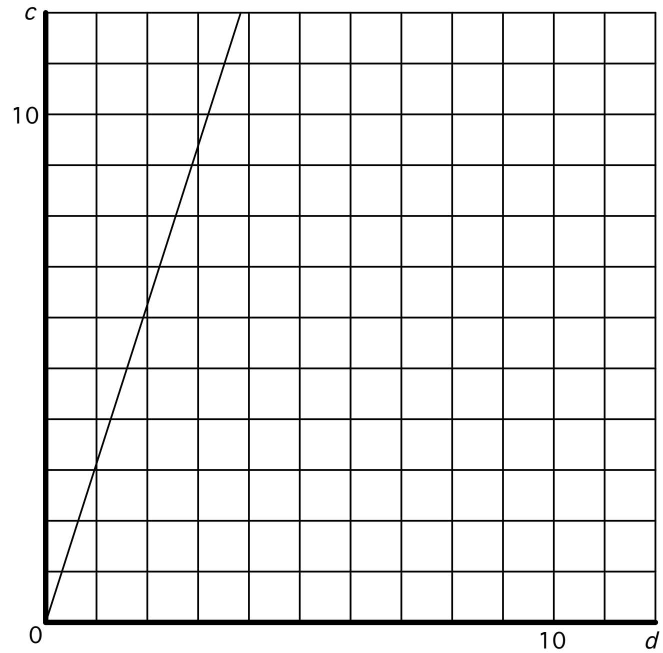
*starting number:* 0

*formula:* $c \approx 3.14d$

*story:*

*d*: diameter of a circle
*c*: circumference of the circle

| $d$ | $c$ |
|---|---|
| 0 | 0 |
| 0.5 | 1.57 |
| 1 | 3.14 |
| 1.5 | 4.71 |
| 2 | 6.28 |
| 2.5 | 7.85 |
| 3 | 9.42 |
| 3.5 | 11.00 |
| 4 | 12.57 |
| 4.5 | 14.14 |
| 5 | 15.71 |




*linear?* no

*rate of change:* - - -

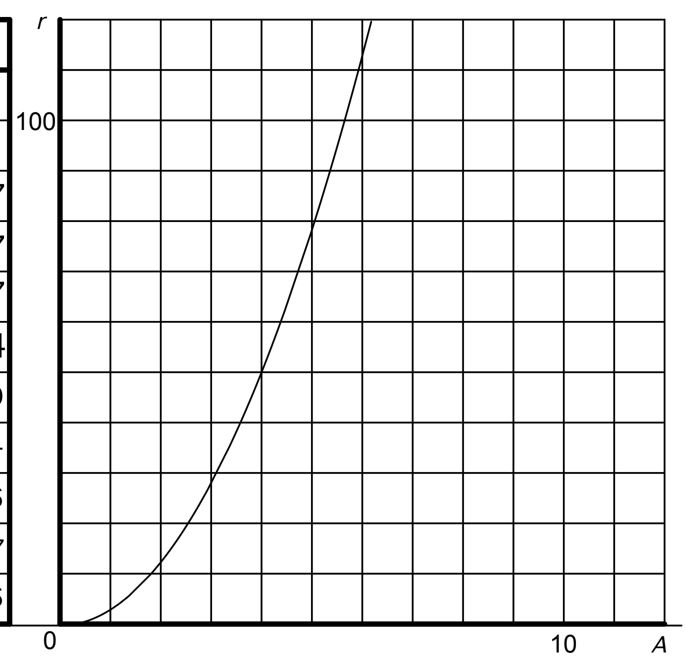
*starting number:* 0

*formula:* $A \approx 3.14r^2$

*story:*

*r*: radius of a circle
*A*: area of the circle

| $r$ | $A$ |
|---|---|
| 0 | 0 |
| 1 | 3.14 |
| 2 | 12.57 |
| 3 | 28.27 |
| 4 | 50.27 |
| 5 | 78.54 |
| 6 | 113.10 |
| 7 | 153.94 |
| 8 | 201.06 |
| 9 | 254.47 |
| 10 | 314.16 |



## Problem #13

*linear?*

*rate of change:*

*starting number:*

*formula:*

*story:*

| | |
|---|---|
| | |
| 3 | 180 |
| | |
| | |
| 7 | 900 |
| | |
| | |
| | |
| | |
| | |
| 20 | 3240 |

## Inverse

*linear?*

*rate of change:*

*starting number:*

*formula:*

*story:*

| | |
|---|---|
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |

## Sample Solutions for #13

*linear?* yes

*rate of change:* 180

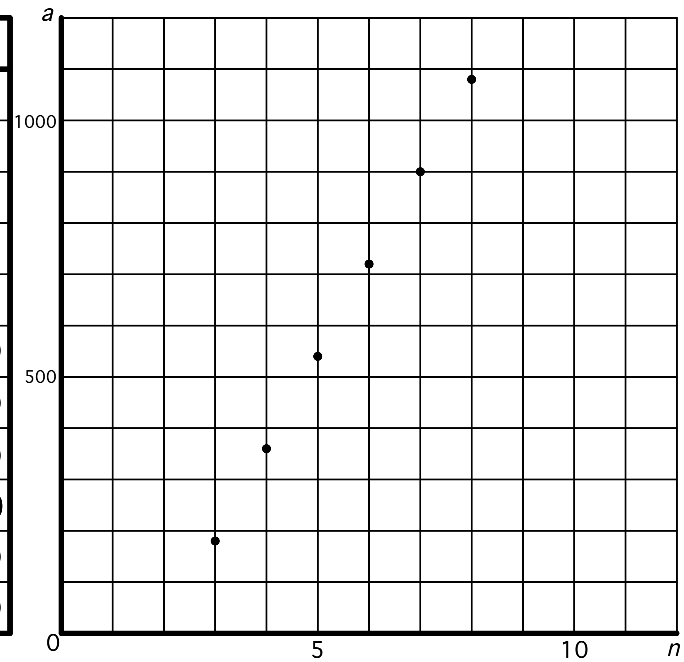
*starting number:* -360

*formula:* $a = 180(n - 2)$ or $a = 180n - 360$

*story:*

$n$: diameter of sides in a polygon
$a$: sum of the interior angles in the polygon

| $n$ | $a$ |
|---|---|
| --- | --- |
| 3 | 180 |
| 4 | 360 |
| 5 | 540 |
| 7 | 900 |
| 8 | 1080 |
| 10 | 1440 |
| 12 | 1800 |
| 14 | 2160 |
| 15 | 2340 |
| 20 | 3240 |




## Inverse

*linear?* yes

*rate of change:* $\frac{1}{180}$

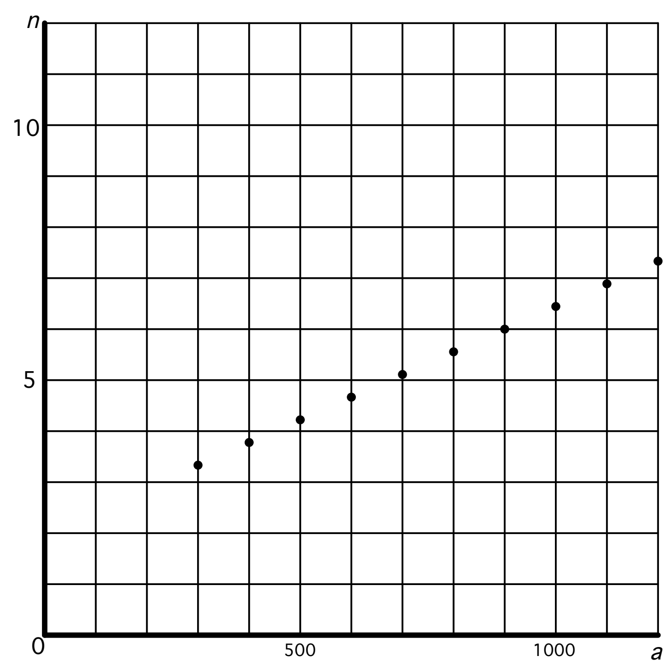
*starting number:* 2

*formula:* $a = \frac{1}{180}n + 2$

*story:*

$a$: sum of the interior angles a the polygon
$n$: diameter of sides in the polygon

| $a$ | $n$ |
|---|---|
| --- | --- |
| 180 | 3 |
| 360 | 4 |
| 540 | 5 |
| 900 | 7 |
| 1080 | 8 |
| 1440 | 10 |
| 1800 | 12 |
| 2160 | 14 |
| 2340 | 15 |
| 3240 | 20 |



## Problem #14

*linear?* yes

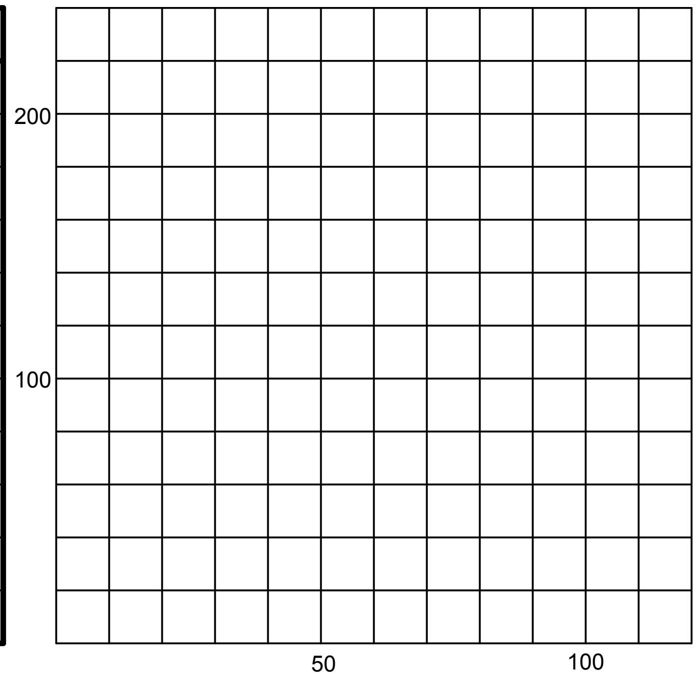
*rate of change:*

*starting number:*

*formula:*

*story:*

| | |
|---|---|
| 0 | 32 |
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| 100 | 212 |




## Inverse

*linear?*

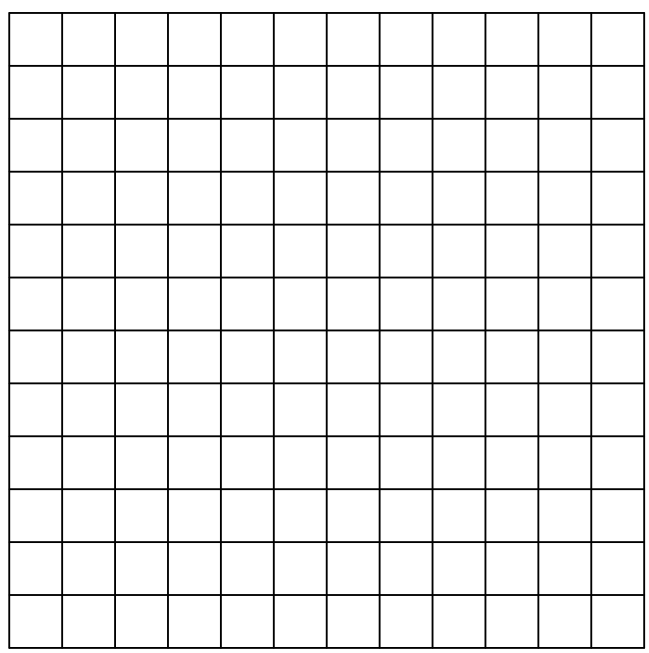
*rate of change:*

*starting number:*

*formula:*

*story:*

| | |
|---|---|
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |



## Sample Solutions for #14

*linear?* yes

*rate of change:* 9/5 or 1.8

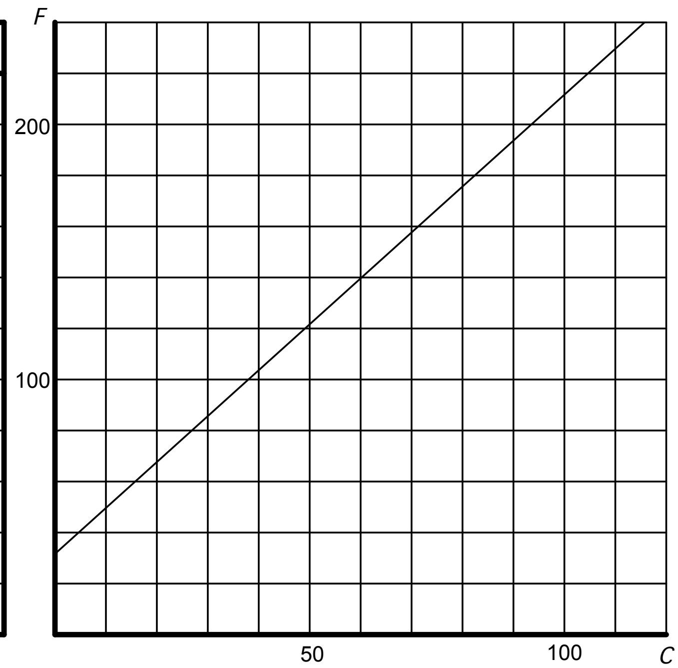
*starting number:* 32

*formula:* $F = 1.8C + 32$

*story:*

$C$: degrees Celsius
$F$: degrees Fahrenheit

| $C$ | $F$ |
|---|---|
| 0 | 32 |
| 10 | 50 |
| 20 | 68 |
| 30 | 86 |
| 40 | 104 |
| 50 | 122 |
| 60 | 140 |
| 70 | 158 |
| 80 | 176 |
| 90 | 194 |
| 100 | 212 |




## Inverse

*linear?* yes

*rate of change:* 5/9

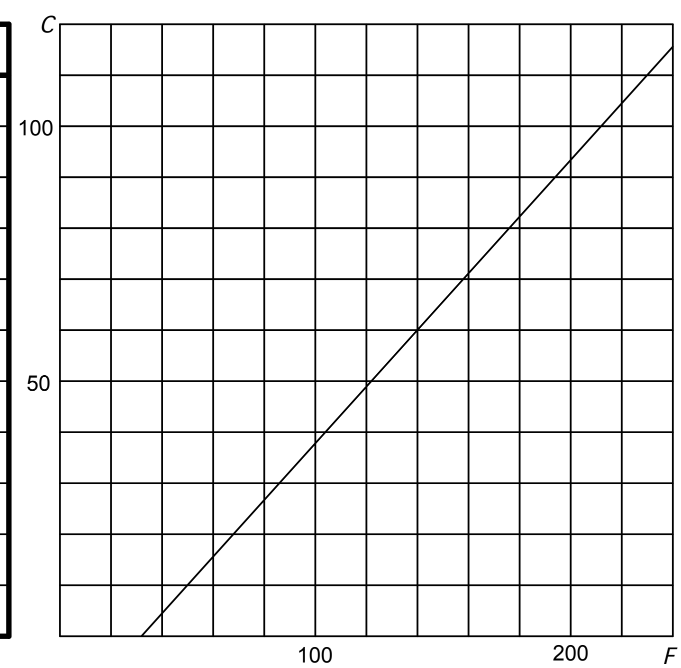
*starting number:* -160/9 or ≈ -17.78

*formula:* $C = (5/9)F - 160/9$
often written as
$C = 5/9(F - 32)$

*story:*

$F$: degrees Fahrenheit
$C$: degrees Celsius

| $F$ | $C$ |
|---|---|
| 32 | 0 |
| 50 | 10 |
| 68 | 20 |
| 86 | 30 |
| 104 | 40 |
| 122 | 50 |
| 140 | 60 |
| 158 | 70 |
| 176 | 80 |
| 194 | 90 |
| 212 | 100 |



## Problem #15

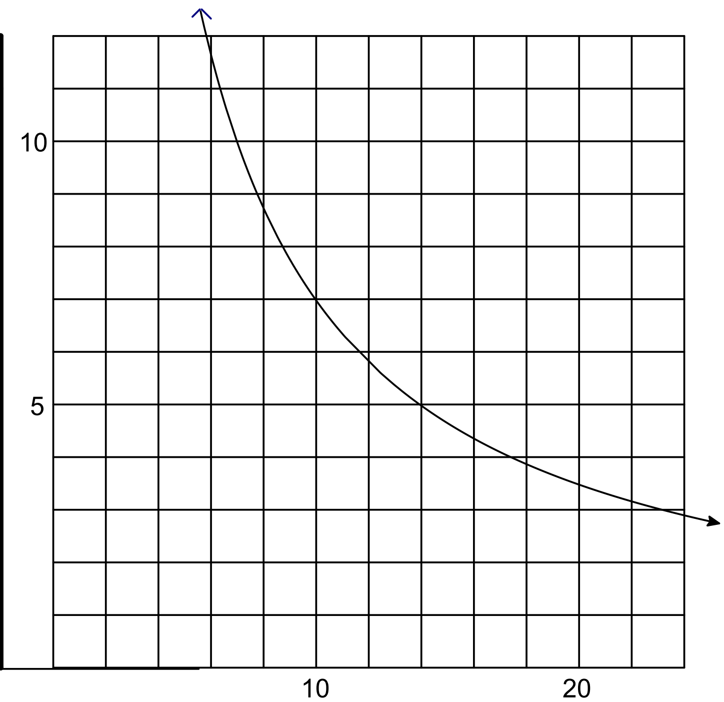
*linear?*

*rate of change:*

*starting number:*

*formula:*

*story:*




## Inverse

*linear?*

*rate of change:*

*starting number:*

*formula:*

*story:*

## Sample Solutions for #15

*linear?* no

*rate of change:* - - -

*starting number:* - - -

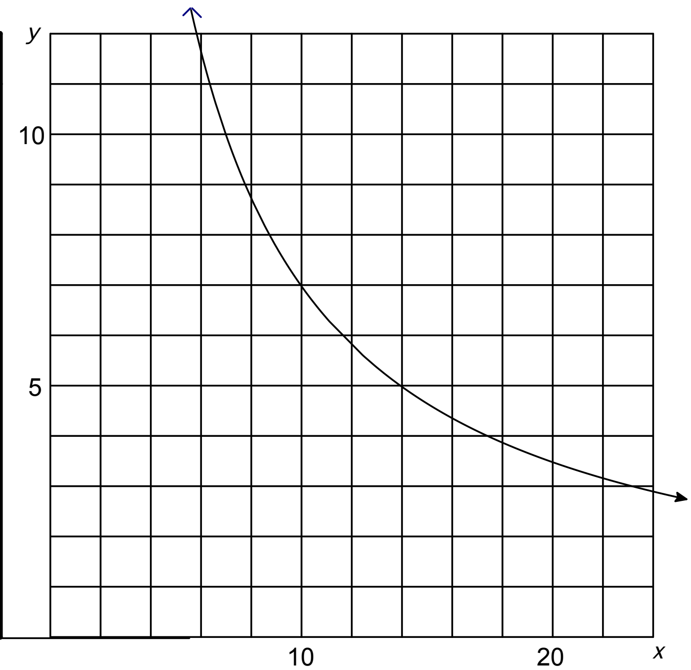
*formula:* $y = 70/x$

*story:*

$x$: interest rate in bank account
$F$: number of years for money to double (approx)

*Note:* The most useful part of the graph for this particular story is between about x = 0 and x = , which is not shown. (It is just above this graph.)

| $x$ | $y$ |
|---|---|
| 0 | - - - |
| 2 | 35 |
| 4 | 17.5 |
| 5 | 14 |
| 7 | 10 |
| 10 | 7 |
| 12 | 5.83 |
| 14 | 5 |
| 16 | 4.38 |
| 18 | 3.89 |
| 20 | 3.5 |




## Inverse

*linear?* no

*rate of change:* - - -

*starting number:* - - -

*formula:* $y = 70/x$

*story:*

$x$: number of years for money to double in bank account (approx)
$F$: interest rate needed to make that happen

*Note:* The most useful part of the graph for this particular story is for y > 5, which is not shown. (It is just off to the right of the graph.)

| $x$ | $y$ |
|---|---|
| 0 | - - - |
| 2 | 35 |
| 4 | 17.5 |
| 5 | 14 |
| 7 | 10 |
| 10 | 7 |
| 12 | 5.83 |
| 14 | 5 |
| 16 | 4.38 |
| 18 | 3.89 |
| 20 | 3.5 |