Deep Algebra Projects: Pre-Algebra/Algebra 1
Capture the Points

Topics

- Solutions and graphs of simple and compound inequalities
- Equivalent inequalities
- Inequalities as they relate to absolute value, exponents, and square roots

The focus of *Capture the Points* is on deepening students' understanding of algebraic inequalities by developing concepts and intuitions—not on learning step-by-step procedures. Students *experiment* instead of practicing pre-taught algorithms. They analyze patterns and relationships between solutions and graphs as they guess, predict, and test values in inequalities.

In order to achieve these goals, the problems ask students to think "backwards." Instead of graphing solutions to inequalities, students construct inequalities that capture or avoid points on a number line. To capture a set of points means to write an inequality for which the points' coordinates are solutions. To avoid a point means to ensure that its coordinates do *not* satisfy the inequality.

Most of the problems ask for two sets of responses: simple and complex. As students search for the simple-looking inequalities, they learn to recognize and make use of patterns between inequalities and various operations. They discover that they can often describe complicated solution sets with relatively simple-looking expressions.

When creating their complex-looking responses, students explore equivalent inequalities in depth. For example, rather than memorizing what happens when multiplying or dividing both sides of an inequality by a negative number, students gradually develop insight into the concept by trying to modify ("complexify") an inequality in ways that do not change its solution set. These experiences will support their later learning around formal methods for solving inequalities.

Even though students are asked to give only a few (usually four) answers per problem, the Solutions include many sample responses. The abundance of responses may look a little overwhelming at first, but it serves three important purposes: (1) it illustrates a variety ways to think about the problem, (2) it enables you to choose your learning goals from among a number of concepts, and (3) it helps you to differentiate for a range of student needs. Certain responses are very challenging to discover, and you may encourage those who need more challenge to search for them.

You may use this activity very flexibly.

- Choose the concepts and operations that you want to focus on.
- Adapt the problems to fit your needs, or create new ones.
- Use it as a written assignment or for classroom lessons.

Regardless of how you approach it, my strongest suggestion is to *provide opportunities for students to share, compare, and critique each other's ideas and examples*—especially the ideas that *don't* work. The number of ideas produced by any one student may be relatively small, but their combined ideas will create excellent opportunities for everyone's learning.

Scoring system

Your job is to write inequalities that capture black (closed) points on a number line and avoid white (open) ones. The scoring system on this page encourages you to write *simple*-looking inequalities. You earn or pay points as outlined below.

Earning points

- Each black point that you capture earns 2 points.
- Capturing all of the black points earns a 5-point bonus.

Paying points

- Each white point that you capture costs 10 points.
- Each connecting word that you use ("and" or "or") costs 2 points.
- Each digit costs 1 point.
- Each appearance of a variable costs 1 point.
- Each "–" (negative or opposite) symbol costs 1 point.
- Each decimal point costs 1 point.

Free

- Inequality symbols are free.
- Operations and fraction bars are free.
- Parentheses or other grouping symbols are free.

Special cases

- Taking a power of a number is free even though it uses digits in the exponent.
- The "opposite" symbol costs 1 point even though "taking the opposite" is a type of operation.

Inequality symbols allowed: <, >, ≤, ≥ Inequality symbol not allowed: ≠

Operations allowed

- Addition, subtraction, multiplication, and division
- Exponents and roots
- Absolute Value

Stage 1

Begin by giving each student a copy of the "Scoring System" handout on the previous page. Discuss the sheet and give examples of calculating scores as needed. Explain that the scoring system applies only to the "Goal 1" inequalities in the Directions.

As indicated in the "What Students Should Know" section below, this activity is designed mainly for students who are familiar with basic concepts of inequalities, but you may also use it as a way of teaching some of these concepts. Depending on students' background, you may provide factual knowledge on an as-needed basis. For example:

(1) A *solution* of an inequality is a number that makes it true.
(2) The *graph* of an inequality is a picture of all of its solutions.
(3) An *interval* is a connected set of points on the number line.
(4) A *closed* or *open* dot at an endpoint of an interval shows that the point *is* or *is not* included in the solution, respectively.

In Problems #1 and #2, students explore linear inequalities. For the Goal 1 responses, they explore different ways to incorporate endpoints into their expressions.

For the more complex expressions (Goal 2), suggest that students begin with a simple-looking inequality from their Goal 1 responses and try to "complexify" it. If they struggle, rather than teaching 'steps,' give them an example or two (without explanation) of pairs of inequalities that have the same solution set. For example:

$$x < 4 \qquad x + 1 < 5,$$
$$x \geq 2 \qquad 3x \geq 6.$$

Let them think about what is happening and how the apply the ideas. Encourage variety. For instance, ask them to create both "greater than" and "less than" examples for the same graph. If they need help, have them compare the solutions sets of a pair of inequalities such as $x > 7$ and either $7 < x$ or $-2x < 14$. All at times, students test their ideas by substituting numbers and checking whether they satisfy the inequality.

Problem #3 opens the door to new concepts and discussions. In order to avoid capturing '0,' students may begin thinking about compound inequalities (involving "and" or "or"), absolute value, exponents, or square roots. Depending on your learning goals, you may choose certain topics to focus on, or you may let students' ideas drive the direction that you take. I do suggest building absolute value into the discussion, because it is a common thread throughout the remaining problems.

Again, depending on your students' background, you may introduce factual information as needed. For example:

(1) The *absolute value of* x ($|x|$) represents the (positive) distance of x from 0.
(2) A *compound inequality* consists of two inequalities joined by "and" or "or."
(3) "And" signifies that a number is a solution only if it satisfies *both* inequalities.
(4) "Or" signifies that a number is a solution if it satisfies *either* inequality.
(5) "And" types of compound inequalities may sometimes be written more compactly in a form such as $a < x < b$, which you may think of as either "$a < x$ and $x < b$" or "x is between a and b."

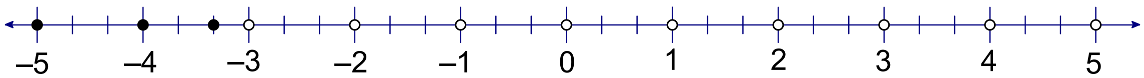
What students should know

- Test the truth of an algebraic inequality for a chosen value of a variable.
- A *solution* of an inequality is a number that makes it true.
- The *solution set* of an inequality is the collection of all numbers that make it true.
- The *graph* of an inequality is a picture of its solution set.
- Understand the meaning of *absolute value*.
- Use closed or open dots at endpoints of an interval to indicate whether a number is included in a solution.

What students may learn (depending on your goals)

- Understand the meanings of "and" and "or" within compound inequalities.
- Using the *absolute value* operation or *squaring* a variable may make some inequalities look simpler by making values positive.
- $|x| = \sqrt{x^2}$.
- You may sometimes eliminate '0' from a solution by putting x in a denominator.
- If you interchange the left and right sides of an inequality, you must change the direction of the inequality symbol in order to keep the solution set the same.
- If you add or subtract the same number on both sides of an inequality, the solution set does not change.
- If you multiply or divide both sides of an inequality by the same positive number, the solution set does not change.
- If you multiply or divide both sides of an inequality by the same negative number, you must change the direction of the inequality symbol in order to keep the solution set the same.
- If you take the reciprocal of both sides of an inequality, you must change the direction of the inequality symbol in order to keep the solution set the same (apart from any values eliminated by having variables in a denominator).

Problem #1



Directions

Capture black points and avoid white points by writing inequalities.

- Write at least two inequalities for each goal.
  - Goal 1: Aim for a high score (for a simple-looking inequality).
  - Goal 2: Use two or more operations (for a more complex inequality).
- Graph each inequality that you write.

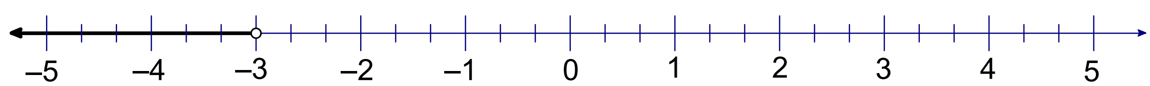
Solutions for #1

General notes for all Solutions

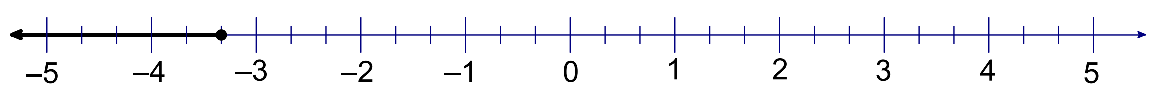
- Remember that students need only two responses for each goal.
- The Solutions contain *sample* answers. Students will have many correct responses that do not appear here, especially for “Two or more operations.”
- Encourage students to be creative. The possibilities are endless!
- Students may occasionally write inequalities that do not capture all of the black points, but the Solutions always show examples that capture all black and avoid all white points.

Aiming for a high score

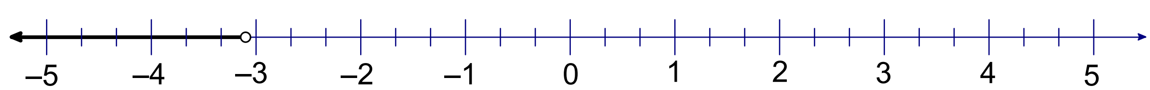
(8 points) $x < -3$



(6 points) $x \leq -3\frac{1}{3}$



(6 points) $x < -3.1$



Example of calculating a score: $x < -3.1$ earns 6 points plus a 5-point bonus for capturing all 3 black dots. The x, –, 3, decimal point, and 1 each cost 1 point (for a total of 5 points). The score is $6 + 5 - 5 = 6$ points.

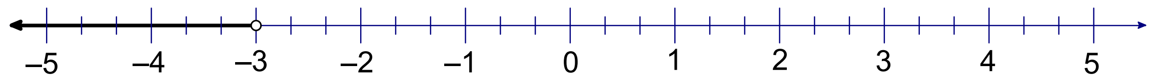
Two or more operations

$2x - 1 < -7$

$4(x + 0.2) < -11.2$

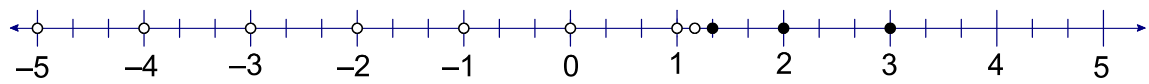
$-3x - 3 > 6$

$2x + 5 < x + 2$



Note: All four of these inequalities have the same graph as $x < -3$, but students may build responses from other graphs as well.

Problem #2



Directions

Capture black points and avoid white points by writing inequalities.

- Write at least two inequalities for each goal.
  - Goal 1: Aim for a high score (for a simple-looking inequality).
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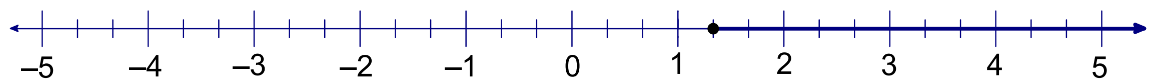
- Graph each inequality that you write.

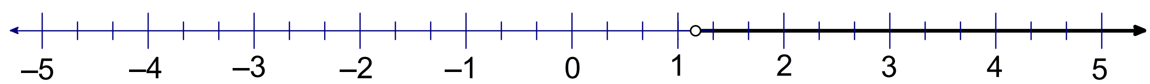
Solutions for #2

Aiming for a high score

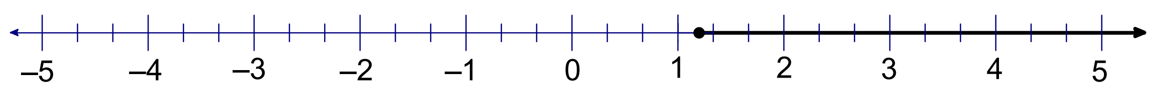
(8 points) $x \geq \frac{4}{3}$



(8 points) $x > \frac{7}{6}$



(7 points) $x \geq 1.2$

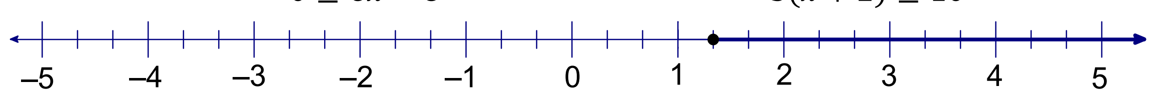


$x > 1.2$ will work equally well. Its graph will have an open dot at 1.2.

Two or more operations

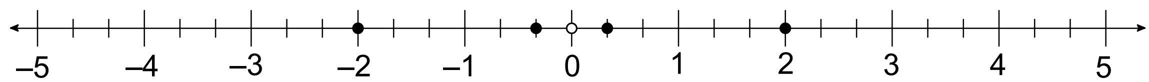
$3x - 4 \geq 0$
$0 \leq 6x - 8$

$-3x + 4 \leq 0$
$3(x + 2) \geq 10$



Note: All four of these inequalities have the same graph as $x \geq \frac{4}{3}$. Students may build their inequalities from other graphs as well.

Problem #3



Directions

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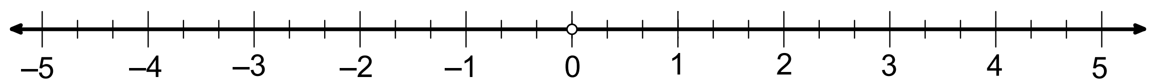
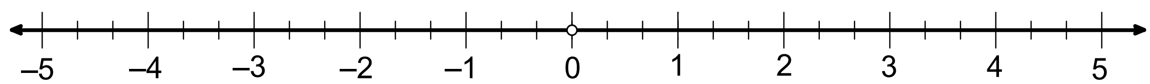
Solutions for #3

Aiming for a high score

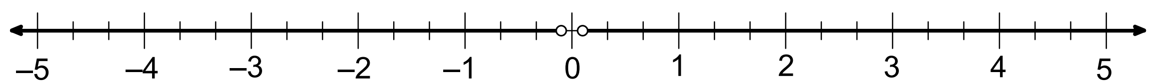
(11 points) $|x| > 0$ $x^2 > 0$ $\sqrt{x^2} > 0$



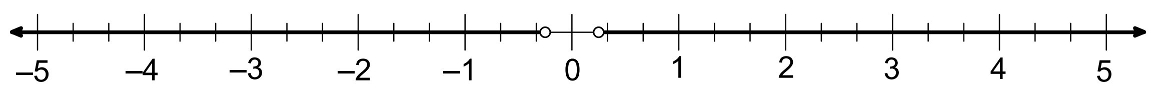
(10 points) $\left|\frac{1}{x}\right| > 0$ $\frac{1}{x^2} > 0$ $1 - |x| < 1$



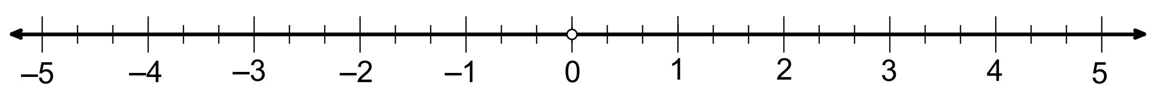
(10 points) $|x| > .1$



(10 points) $\sqrt{|x|} > .5$



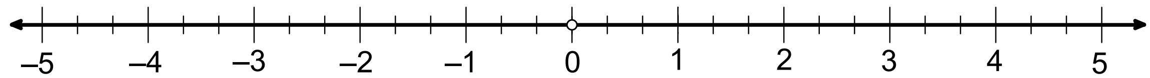
(7 points) $x > 0$ or $x < 0$



This response doesn't give a great score, but it is a natural way to capture the points.

Three or more operations

$0 < \left|\frac{6x+7}{x}\right|$ $\left(\frac{3}{x^2}+5\right)^2 > 25$



Students may build complex inequalities from other graphs as well.

Stage 2

Throughout this project, students gradually build the knowledge and tools needed to handle increasingly complex graphs:

Problem #4	An interval that is (at least potentially) symmetric across ‘0’
Problem #5	An extension of Problem #3 in which the missing point is shifted away from 0
Problem #6	An interval that is not symmetric across ‘0’
Problem #7	An interval that is not symmetric across ‘0’ and uses “messier” coordinates

Notice that each solution in Problem #5 is a modified version of a corresponding solution in Problem #3. In every case, x is replaced by $x - 2$ in order to shift the solution set 2 units to the right. In general, you will see that solutions to one problem are often modified versions of solutions to other problems. The purpose is to emphasize patterns and relationships. If necessary, you may eventually lead students to discover these patterns. However, they may have many other approaches, and you should always allow them to take their own paths, at least initially.

A couple of notes about specific types of solutions:

- Compound inequalities appear later in each Solution page, because they earn fewer points. However, you may consider them important to discuss.
- You may represent all of the solutions sets by polynomial inequalities. However, the Solutions do not mention polynomials explicitly until Problem #9, because doing so would interrupt the flow of the other concepts. If you choose to follow up on the polynomials idea, you and your students will need to construct solutions yourselves. Students may find it more manageable *and* enlightening to write them in factored form.

What students should know

- Understand the concepts from Stage 1.

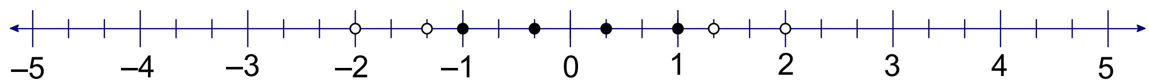
What students may learn (depending on your goals)

- Reinforce and extend concepts from Stage1.
- Understand and translate between different forms of compound inequalities. For example:

$$x > a \text{ and } x < b \quad \leftrightarrow \quad a < x < b$$

- Discover methods for using absolute value (understanding it in terms of distance) to write inequalities for intervals on the number line.
- Discover and apply methods for translating solutions sets to the left or right on the number line.
- Begin to discover techniques for modifying inequalities in order to eliminate isolated numbers from a solution set.

Problem #4



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- Write at least two inequalities for each goal.
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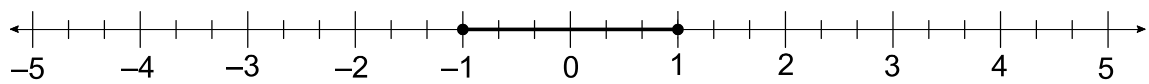
- Graph each inequality that you write.

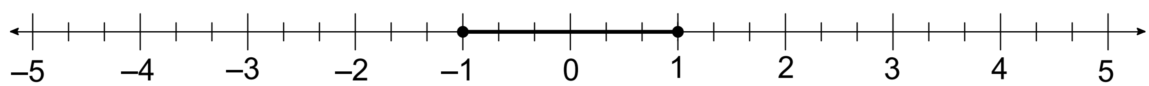
Solutions for #4

Aiming for a high score

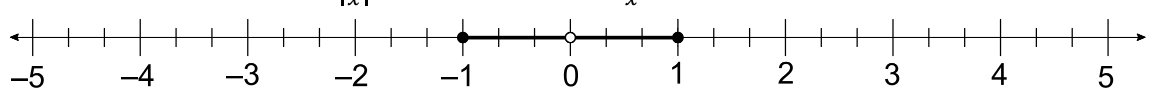
(11 points) $|x| \leq 1$ $x^2 \leq 1$ $\sqrt{x^2} \leq 1$



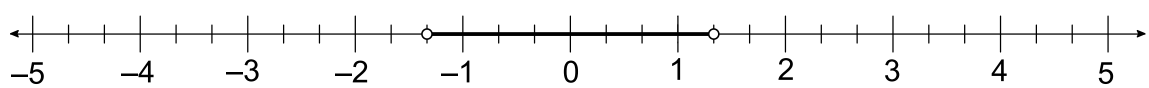
(10 points) $1 - |x| \geq 0$



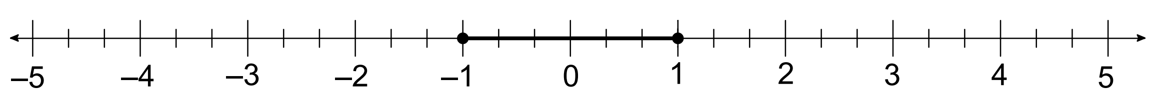
(10 points) $\left|\frac{1}{x}\right| \geq 1$ $\frac{1}{x^2} \geq 1$



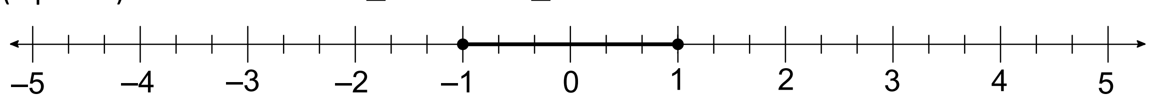
(10 points) $|x| < \frac{4}{3}$



(9 points) $-1 \leq x \leq 1$



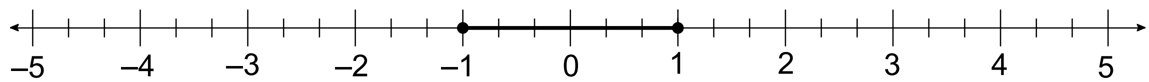
(6 points) $x \geq -1$ and $x \leq 1$



This response doesn't give a great score, but it's a natural way to capture the points.

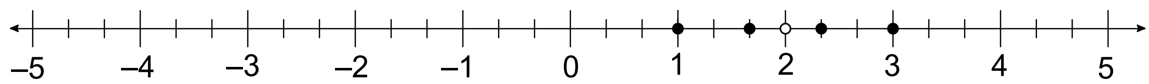
Two or more operations (Four of the inequalities above satisfy this condition as well!)

$2 \leq \frac{x+5}{2} \leq 3$ $2x + 7 \geq 5$ and $9(x + 10) \leq 99$



Students may build complex inequalities from other graphs as well.

Problem #5



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Capture black points and avoid white points by writing inequalities.

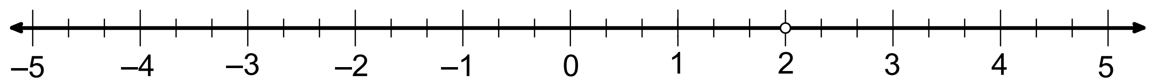
- Write at least two inequalities for each goal.
  - Goal 1: Aim for a high score (for a simple-looking inequality).
  - Goal 2: Use three or more operations (for a more complex inequality).
- Graph each inequality that you write.

Solutions for #5

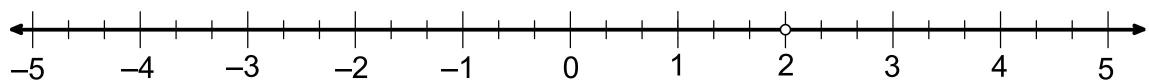
Notes: In each inequality below, the variable x from the corresponding solution in Problem #3 was replaced by $x-2$. Guide students to discover that (and to understand why) this has the effect of shifting each solution set 2 units to the right. (See the Introduction for Stage 2.) Of course, many other approaches and solutions are possible.

Aiming for a high score

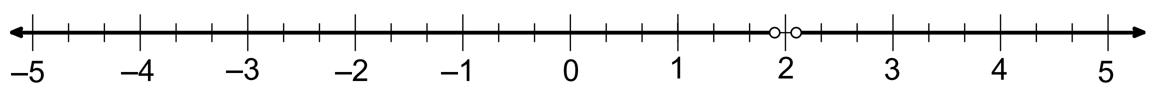
(10 points) $|x-2|>0$ $(x-2)^2>0$ $\sqrt{(x-2)^2}>0$



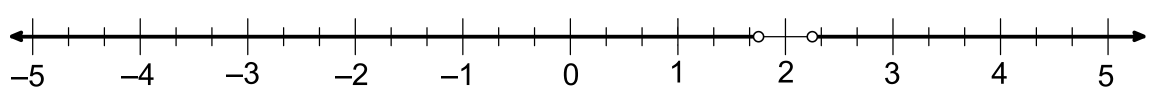
(9 points) $\left|\frac{1}{x-2}\right|>0$ $\frac{1}{(x-2)^2}>0$ $1-|x-2|<1$



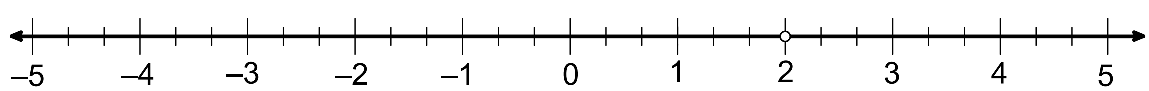
(9 points) $|x-2|>.1$



(9 points) $\sqrt{|x-2|}>.5$



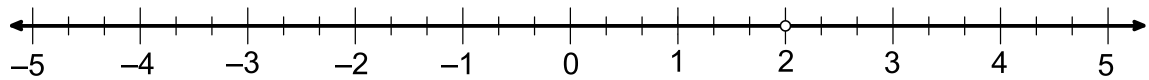
(7 points) $x>2$ or $x<2$ (5 points) $x-2>0$ or $x-2<0$



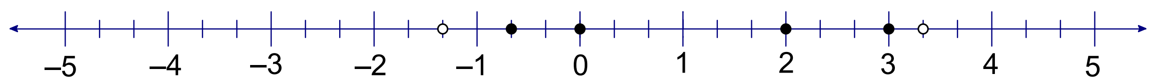
These responses don't give great scores but are natural ways to capture the points.

Three or more operations

$$0 < \left|\frac{6(x-2)+7}{x-2}\right| \qquad \left(\frac{3}{(x-2)^2}+5\right)^2 > 25$$



Problem #6



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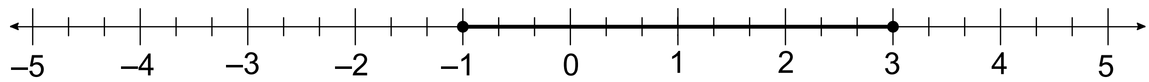
Capture black points and avoid white points by writing inequalities.

- Write at least two inequalities for each goal.
  - Goal 1: Aim for a high score (for a simple-looking inequality).
  - Goal 2: Use two or more operations (for a more complex inequality).
- Graph each inequality that you write.

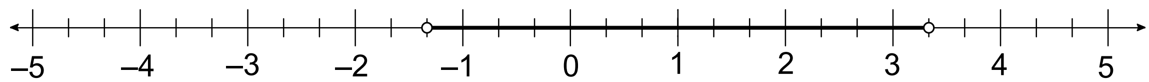
Solutions for #6

Aiming for a high score

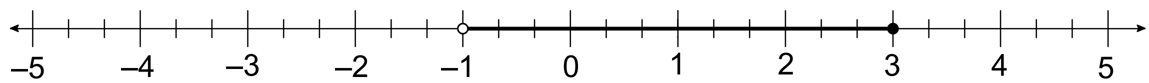
(10 points) $|x-1| \le 2$ $(x-1)^2 \le 4$ $\sqrt{(x-1)^2} \le 2$



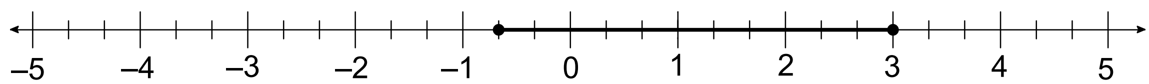
(9 points) $|x-1| < \frac{7}{3}$



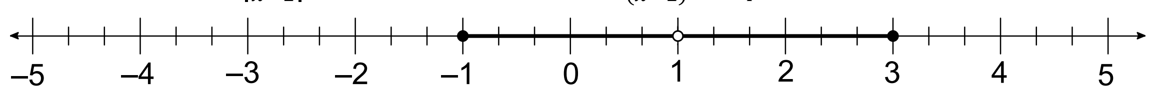
(9 points) $-1 < x \le 3$



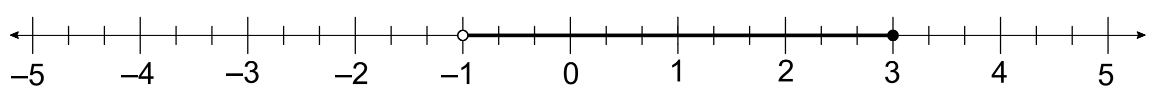
(8 points) $-\frac{2}{3} \le x \le 3$



(8 points) $\left|\frac{1}{x-1}\right| \ge .5$ $\frac{1}{(x-1)^2} \ge \frac{1}{4}$

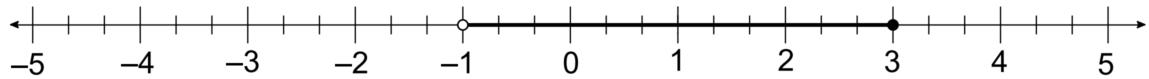


(6 points) $x > -1$ and $x \le 3$



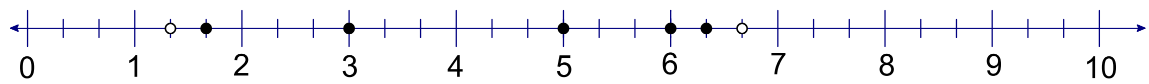
Two or more operations (Many of the inequalities above satisfy this condition as well!)

$2 < \frac{x+5}{2} \leq 4$ $\quad$ $2x + 7 > 5$ and $9(x + 8) \leq 99$



Students may build complex inequalities from other graphs as well.

Problem #7



Directions

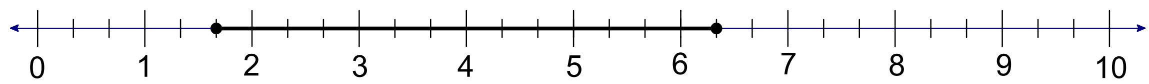
Capture dark points and avoid light points by writing inequalities.

- Write two inequalities for each goal.
  - Goal 1: Aim for a high score (for a simple-looking inequality).
  - Goal 2: Use two or more operations (for a more complex inequality).
- Graph each inequality that you write.

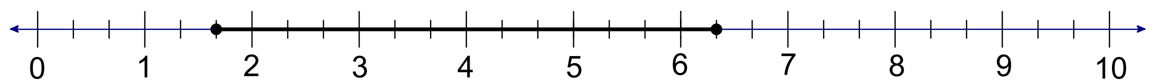
Solutions for #7

Aiming for a high score

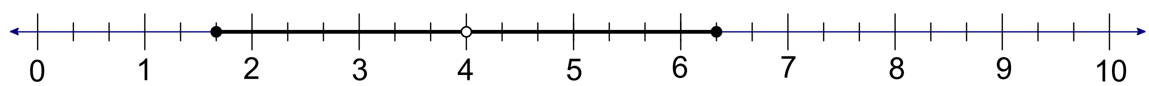
(11 points) $|x-4| \leq \frac{7}{3}$ $\sqrt{(x-4)^2} \leq \frac{7}{3}$



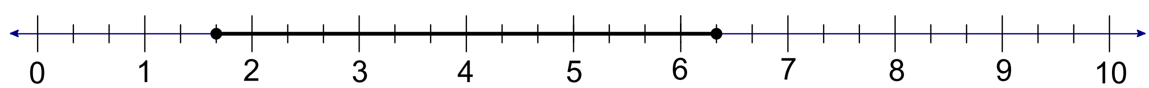
(10 points) $(x-4)^2 \leq \frac{49}{9}$



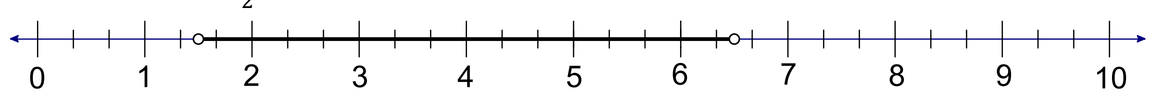
(10 points) $\left|\frac{1}{x-4}\right| \geq \frac{3}{7}$ $0 < |x-4| \leq \frac{7}{3}$ (9 points) $\frac{1}{(x-4)^2} \geq \frac{9}{49}$



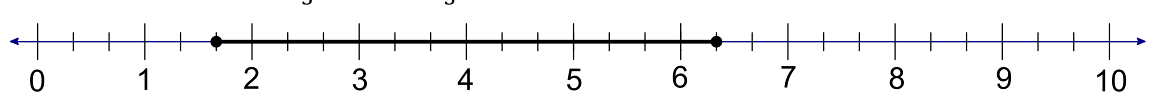
(9 points) $\frac{5}{3} \leq x \leq \frac{19}{3}$ $\frac{5}{3} \leq x \leq 6\frac{1}{3}$



(9 points) $\frac{3}{2} < x < 6.5$



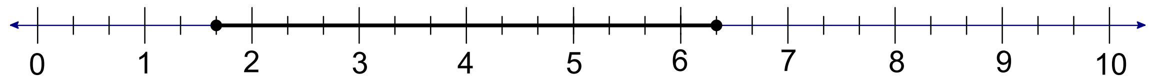
(6 points) $x \geq \frac{5}{3}$ and $x \leq \frac{19}{3}$



Two or more operations (Many of the inequalities above satisfy this condition as well!)

$\frac{10}{3} \leq \frac{x+5}{2} \leq \frac{17}{3}$

$2x + 7 \geq 10\frac{1}{3}$ and $9(x - 6) \leq 3$



Stage 3

In Stage 3, students continue to build new knowledge and tools needed to handle increasingly complex graphs:

Problem #8 Two intervals that are symmetric across ‘0’
Problem #9 Two intervals that are not symmetric across ‘0’

“Goal 2” (using two or more operations) is omitted, because the focus is on creating inequalities that look as concise as possible.

In Problem #8, the Solutions do not show the usual large range of possibilities. However, you and your students may still choose to explore them!

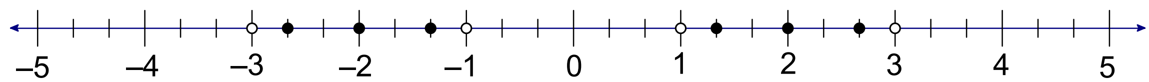
What students should know

- Understand the concepts from Stages 1 and 2.

What students may learn (depending on your goals)

- Reinforce and extend concepts from Stages 1 and 2.
- Discover concise methods of representing solution sets that consist of more than one interval.
- Discover that replacing x by $|x|$ in an inequality unites the original solution set with its reflection across ‘0.’

Problem #8



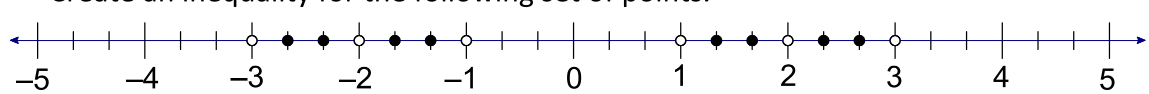
Directions

Capture black points and avoid white points by writing inequalities.

- Write two inequalities. Aim for a high score.
- Graph each inequality that you write.

Diving Deeper

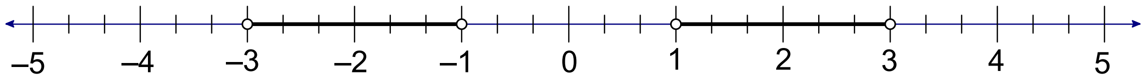
Create an inequality for the following set of points.



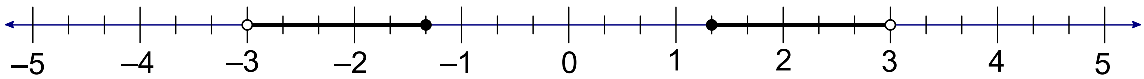
Solutions for #8

Aiming for a high score

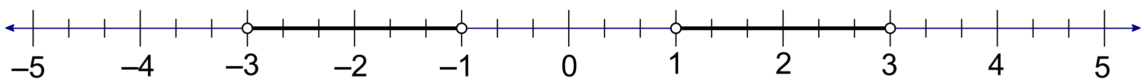
(14 points) $1 < |x| < 3$ $1 < x^2 < 9$ $\big||x| - 2\big| < 1$



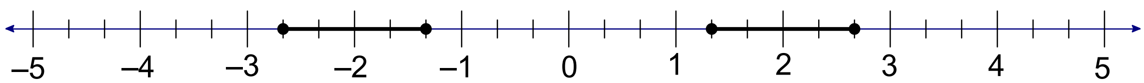
(13 points) $\frac{4}{3} \le |x| < 3$ (12 points) $\frac{16}{9} \le x^2 < 9$



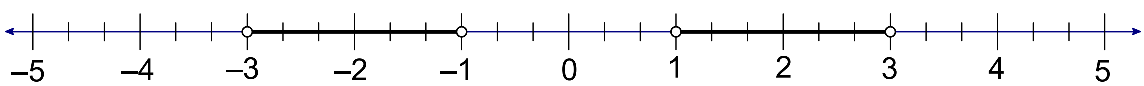
(12 points) $\frac{1}{3} < \left|\frac{1}{x}\right| < 1$ $\frac{1}{9} < \frac{1}{x^2} < 1$ $1 > \frac{1}{x^2} > \frac{1}{9}$



(12 points) $\frac{4}{3} \le |x| \le \frac{8}{3}$ (10 points) $\frac{16}{9} \le x^2 \le \frac{64}{9}$

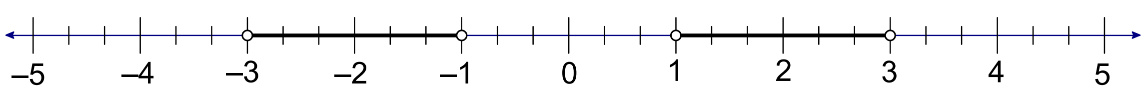


(9 points) $|x - 2| < 1$ or $|x + 2| < 1$



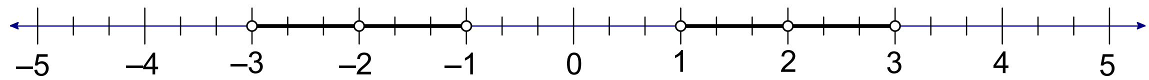
It may be acceptable (but not quite clear?) to write $|x \pm 2| < 1$, which would be worth 14 points!

(1 point) $(x > -3$ and $x < -1)$ or $(x > 1$ and $x < 3)$

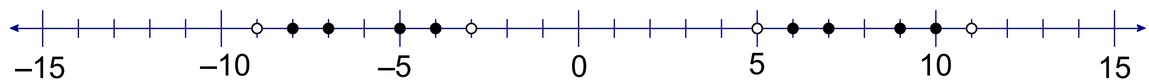


This solution has a poor point value, but it is definitely worth discussing!

*A 17-point solution for **Diving Deeper**:* $0 < \big||x| - 2\big| < 1$



Problem #9



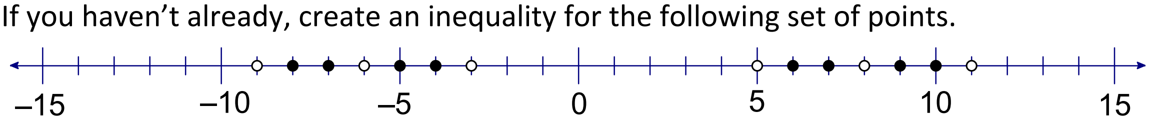
Directions

Capture black points and avoid white points by writing inequalities.

- Write at least one inequality. Aim for a high score.
- Graph each inequality that you write.
- Explain your thinking process.

Diving Deeper

If you haven't already, create an inequality for the following set of points.



Solutions for #9

Aiming for a high score

(17 points) $\left||x-1|-7\right| \leq 2$

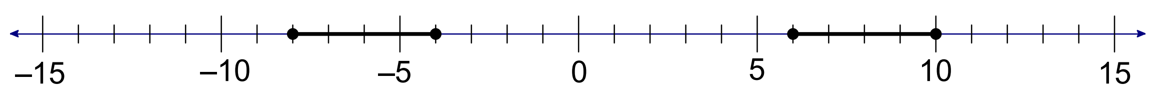
Look below for a reasoning process.

(11 points) $(x+8)(x+4)(x-6)(x-10) < 0$

I have not suggested the idea of using factored polynomials until now, but you could apply it to most of the problems in this project. An advantage of this approach is the ease of dealing with intervals of different sizes within the same graph. Of course, you would need to have very different kinds of conversations with students.

(10 points) $-8 \leq x \leq -4$ and $6 \leq x \leq 10$

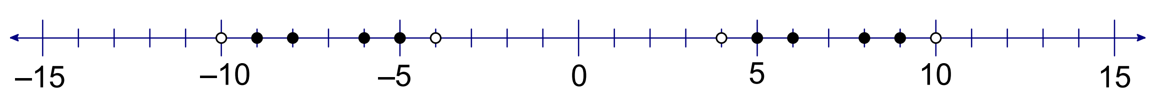
All three of these solutions share the same graph.



You may build solutions from many other graphs as well.

A thinking process for $\left||x-1|-7\right| \leq 2$

(1) Shift the graph 1 unit to the left in order to make it symmetrical across $x = 0$.



(2) Capture the points on the right (from 5 to 9 including the endpoints).
The center of the region is at 7, and it extends a distance of 2 on either side.
Therefore, an inequality that captures the points is

$$|x-7| \leq 2.$$

(3) Replace x by $|x|$ in order to include the symmetric region across $x = 0$.

$$\left||x|-7\right| \leq 2.$$

(4) Return the solution set to its original position. That is, shift it 1 unit to the right by subtracting 1 from x.

$$\big||x-1|-7\big| \leq 2.$$

The Diving Deeper question

In order to eliminate the midpoint from each interval, ensure that the algebraic expression does not equal 0.

$$0 < \big||x-1|-7\big| \leq 2$$

You may check that 8 and –6 now fail to make the inequality true.