Deep Algebra Projects: Algebra 1 / Algebra 2
Latrice's Greeting Cards

Topics

- Linear equations and functions (review and apply)
- Solving optimization problem involving quadratic functions
- Factoring quadratic expressions
- Solving quadratic equations numerically, graphically, and symbolically
- Analyzing graphs of quadratic functions
- Vertex form of parabolas (and completing the square)
- Connecting the quadratic formula to graphs of quadratic functions

In Latrice's Greeting Cards, students explore a business application of quadratic functions from multiple perspectives. Rather than working from a ready-made quadratic expression for profit, students construct the function themselves after analyzing the relationship between the price and demand for a product.

Students begin by exploring the situation numerically. As they progress to equations and graphs, they gain new insights and find more efficient approaches, culminating in a more general analysis of the situation in Stage 3.

You may use this project as a supplementary activity in which students consolidate and apply previously learned knowledge about quadratic functions. Alternatively, you may integrate the project into your plans to *teach* key concepts, especially in Stage 2. The latter approach may be more challenging and meaningful for students, because it requires them to develop many concepts and methods for themselves *before* they receive explicit instruction on the relevant algebraic procedures. Watch for more detailed guidance in the introduction to Stage 2.

Note: Each problem begins with a Thinking Prompt page and a Problem page. You may use either or both of these. The Thinking Prompt is more open-ended; it encourages students to make observations and ask questions before they receive concrete directions. Typically, students will anticipate many of the questions on the Problem page or come up with important ideas or questions of their own. Follow up on the questions that students create! In fact, you may collaborate with them to modify (or even ignore!) directions on the Problem page as you strive to best respect their personal learning needs and interests. Think of the entire project as a suggested framework for an investigation rather than a strict set of instructions.

Stage 1

In Stage 1, students begin analyzing a business scenario in which an entrepreneur, Latrice, is deciding upon the optimum price for her product (that is, the price that results in the greatest profit). Depending on your students' backgrounds and thinking styles, they may approach the tasks in many different ways. Do not feel limited by the sample approaches shown in the Solutions. Focus on your learning goals and your *students'* ideas, and build from there.

Note: In order to make the equations reflect the practical meaning of the problems, I often use variable names that fit the context. For example, in place of $y = mx + b$ for the slope-intercept form of the equation of a line, I use $D = mP + n$. The input and the output are the price (P) and the demand (D), respectively. I use the usual m for the slope, but I use n in place of b for the y-intercept in order to avoid later confusion with the coefficient, b, in the standard form of the quadratic expression, $ax^2 + bx + c$. Using non-traditional variable names may initially feel confusing for some students. However, it has the advantage of focusing their attention on the mathematical structure of an equation as opposed to the particular symbols.

What students should know

- Understand linear functions (including slopes and y-intercepts) numerically, graphically, and symbolically.
- Find algebraic formulas for tables that show linear patterns.
- Solve linear equations.
- Multiply and factor binomials and trinomials.

What students will learn

- Apply knowledge of linear functions to optimize profit for a business.
- Think critically about the nature and source of a set of data.
- Explore the mathematical and practical effects of multiplying two linear functions.
- Informally find or estimate the maximum value of a quadratic function using numerical and graphical approaches.

Thinking Prompt #1

What do you notice? What do you wonder?

Latrice is starting a business making and selling greeting cards.
It costs her about $2.50 in materials to make each card.

Projections

Price per card dollars	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00	5.25	5.30
Demand number per year	1000	920	840	760	680	600	520	440	360	270	160	40	0

Problem #1

Projections

Price per card dollars	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00	5.25	5.30
Demand number per year	1000	920	840	760	680	600	520	440	360	270	160	40	0

Cost to make each card: $2.50

Directions

- Look for and describe patterns in the table.
- Graph the data. Explain how your graph shows the patterns.
- Explain what might cause the patterns and what they mean for Latrice's business.
- Imagine where Latrice might have obtained these projections.
- Help Latrice decide on a selling price for her cards. Explain your reasoning.

Conversation Starters for #1

Thinking Prompt

I wonder what kinds of greeting cards Latrice makes.

> The cards could be for birthdays, holidays, congratulations, thanks, sympathy, etc. Students may have their own creative ideas about the kinds of cards that people might be interested in.

I wonder what that the title "Projections" means.

> Projections are p*redictions* about what will happen (not outcomes that have already happened).

I wonder how realistic these projections are likely to be.

I wonder if there is a pattern to the projections.

I notice that Latrice does not make any money at the extreme ends of the table.

> On the left, the price is too low to cover her costs. On the right, the price is so high that people will not buy the cards.

I wonder if an entrepreneur would normally be able to obtain projections like these.

The Problem

I wonder what *demand* means.

> Demand is used in business to describe the degree to which some product or service is wanted by consumers. In this case, it represents the number of cards that Latrice sells each year.

I wonder what kind of graph would be the best for showing the data.

I notice that the price is going up 25¢ at a time (until the very last column).

I notice that the demand is going down at a steady rate until the last few columns.

I notice that Latrice's business will probably be pretty small at first.

I notice a trade-off in setting the price. (When the price goes up, fewer people buy.)

I predict that the ideal price is somewhere in the middle of the table.

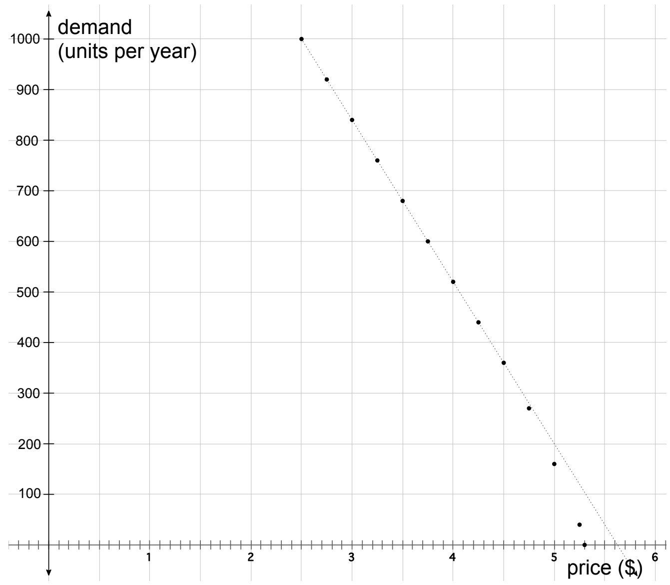
I wonder if I need to take account of the cost in order to determine the optimum selling price.

Sample Solutions for #1

Patterns in the table

For prices between $2.50 and $4.50, the demand decreases at a steady 80 units (greeting cards) per 25¢ increase in price. For prices greater than $4.50, the demand decreases more quickly, and the rate is no longer constant.

Graph



The graph drops in a straight line at first, showing that the demand is decreasing at a steady rate as the price increases. In the rightward part of the graph, the dots fall below the line defined by the other points, showing that the demand decreases more quickly once the price reaches a certain level.

Interpreting the patterns

People are less willing to buy the greeting cards as the price increases. Demand falls at a constant rate until the price reaches a certain point for which the effect on demand is even greater. Eventually, when the price reaches $5.30, people are no longer willing to buy the cards.

Where Latrice might have obtained the projections

Prospective business owners often ask outside experts to do a market analysis in order to help them create a business plan. This analysis may include a prediction of the demand-price relationship. It may or may not be practical for Latrice to do this for such a small business, but we will assume that she can in order that we can explore the mathematical ideas involved!

Choosing the best price for the cards

The best price is probably about \$4.00 per card.

Latrice can determine her *profit* per card (the net amount she makes after accounting for her costs) by subtracting \$2.50 (the amount it costs to make each card) from the selling price of each card and then multiplying the result by the number of cards she sells.

selling price per card – cost to make a card = profit per card
profit per card · number of cards sold = total profit

$$
\begin{aligned}
(2.50 - 2.50) \cdot 1000 &= 0 \cdot 100 = 0 \\
(2.75 - 2.50) \cdot 920 &= 0.25 \cdot 920 = 230 \\
(3.00 - 2.50) \cdot 840 &= 0.50 \cdot 840 = 420 \\
(3.25 - 2.50) \cdot 760 &= 0.75 \cdot 760 = 570 \\
(3.50 - 2.50) \cdot 680 &= 1.00 \cdot 680 = 680 \\
(3.75 - 2.50) \cdot 600 &= 1.25 \cdot 600 = 750 \\
(4.00 - 2.50) \cdot 520 &= 1.50 \cdot 520 = 780 \\
(4.25 - 2.50) \cdot 440 &= 1.75 \cdot 440 = 770 \\
(4.50 - 2.50) \cdot 360 &= 2.00 \cdot 360 = 720 \\
(4.75 - 2.50) \cdot 270 &= 2.25 \cdot 270 = 607.50 \\
(5.00 - 2.50) \cdot 160 &= 2.50 \cdot 160 = 400 \\
(5.25 - 2.50) \cdot 40 &= 2.75 \cdot 40 = 110 \\
(5.30 - 2.50) \cdot 0 &= 2.80 \cdot 0 = 0
\end{aligned}
$$

The best price appears to be about \$4.00 per card, because it results in the greatest profit (\$780).

Notes:

- Students may observe that the profit increases more and more slowly until it reaches a peak and then begins to fall, first slowly and then more quickly. (In fact, this change follows a specific pattern of its own that students may explore!)
- Some students may simply calculate the income without taking account of the costs. This approach is fine for finding the best price per card, because the cost for each card is the same.
- Some students may suggest an optimal price slightly greater than \$4.00, because *increasing* the price \$0.25 beyond \$4.00 reduces the profit by less (\$10) than *decreasing* the price by \$0.25 (\$30). A few may even calculate profits for prices between \$3.75 and \$4.25 (by using the linear pattern for demand). If they do this cent by cent, they may discover that \$4.06 appears to give the greatest profit, though the improvement over the \$4.00 price is very small (only about \$1.25 per year!). Since the demand values are approximate anyway, \$4.00 is probably a reasonable choice.

Thinking Prompt #2

What do you notice? What do you wonder?

Equations and graphs can help you see more deeply into patterns.

Projections

Price per card dollars	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50
Demand number per year	1000	920	840	760	680	600	520	440	360

Cost to make each card: $2.50

Problem #2

Projections

Price per card dollars	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50
Demand number per year	1000	920	840	760	680	600	520	440	360

Cost to make each card: $2.50

Directions

- Write an algebraic expression or equation that shows the relationship between price and demand.
- Write an algebraic expression or equation that shows the relationship between price and profit.
- Graph the price-profit relationship.
- Explain how the graph illustrates the optimum selling price for Latrice's cards.
- Explain or show how your graph would be affected if you included the data from the final four columns in Problem #1.

Conversation Starters for #2

Thinking Prompt

I notice that the final four columns from Problem #1 are missing. I *wonder* why.

These four columns do not fit the pattern in the rest of the table. It may be easier to begin by focusing on the pattern before thinking of how to fit these columns into the picture.

I wonder if I can find an equation to fit the table.

I wonder if the equation could help me find the optimum selling price.

I wonder if it is possible to find a single equation that would fit *all* of the numbers from the table in Problem #1.

The Problem

I notice that the price-demand relationship is linear. (It has a constant rate of change.)

I notice that the P-intercept is not included in the table. *I wonder* what is the best way to find it.

I notice that some of my data from Problem #1 help me graph the profit function.

I notice that the price-profit relationship is not linear.

I notice that the points suggest a smooth curve that rises (from left to right), then falls.

I notice that the optimum price should be near the top of the graph.

I wonder if the highest point that I plotted is at the very peak (vertex) of the graph.

I notice that I can identify the highest point by plotting more points near the top.

I wonder how I can find an equation for a relationship that is not linear.

Develop your own method by thinking about the calculations that you did with the numbers in Problem #1.

I notice different ways to write the profit equation. *I wonder* if it matters which I use.

I notice that I can test my equation by seeing if it gives the correct values for the profit.

Students may again refer to their calculations from Problem #1.

Sample Solutions for #2

The relationship between price and demand

$$D = -320P + 1800$$

P is the price of a card.

D is the demand (number of cards sold per year).

Students may have different approaches to finding this equation. Some may use a trial and error process with the data in the table. Others may approach the task algebraically.

Strategy 1

Find the slope and the *y*-intercept (in this case, the D-intercept).

The slope, m, is the rate of change. Choosing two pairs from the table:

$$m = \frac{520 - 840}{4.00 - 3.00} = -320$$

The value, –320, indicates that for each increase of $1.00 in price, the demand decreases by 320 cards.

The D-intercept is the value of D at which $P = 0$. Some students may find it by using proportional reasoning to trace the table backwards to $P = 0$. Others may use the slope-intercept form of the equation of a line, $y = mx + b$ (which we write as $D = mP + n$). They will need to choose one ordered pair from the table. I have chosen (2.50, 1000).

$$D = mP + n$$
$$1000 = -320(2.50) + n$$
$$1000 = -800 + n$$
$$1800 = n$$

Substituting the values of m and n into $D = mP + n$, the result is

$$D = -320P + 1800$$

Strategy 2

Use the point-slope form of the equation of a line.

$y - y_1 = m(x - x_1)$ becomes $D - D_1 = m(P - P_1)$ where (P_1, D_1) is a particular point in the table. Again, choosing the point (2.50, 1000):

$$D - 1000 = -320(P - 2.50)$$

$$D - 1000 = -320P + 800$$
$$D = -320P + 1800$$

Other approaches are possible.

The relationship between price and profit

$$F(P) = (P - 2.50)(-320P + 1800)$$
$$= -320P^2 + 2600P - 4500$$

F is the profit per year. P is the price of each card.

A possible thinking process

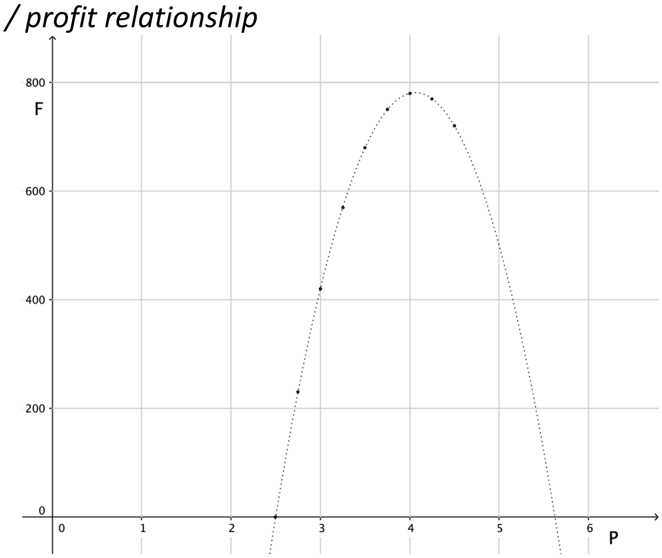
The profit is the income (revenue) minus the cost of making the cards.

total income = price of each card · number of cards sold = PD
total cost = cost to make each card · number of cards made = CD
profit = total income – total cost = $PD - CD = (P - C)D$

$$F(P) = (P - C)D$$
$$F(P) = (P - 2.50)(-320P + 1800)$$
or
$$F(P) = -320P^2 + 2600P - 4500$$

These formulas assume that Latrice makes the same number of cards that she sells. They do not take into account any additional expenses that Latrice may face.

Graph of the price / profit relationship

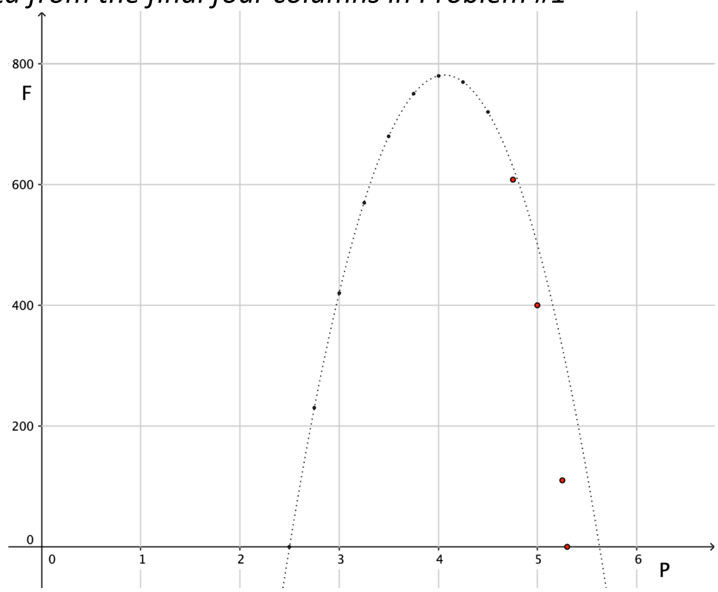


The dots show points for the P values in the table for *this problem*. Larger values of P from the table in Problem #1 are not yet included. Some students may sketch a smooth curve (in this case, a parabola) through these points (see the dotted line).

How the graph illustrates the optimum selling price for Latrice's cards

The highest point on the graph represents the ideal selling price, because it corresponds to the maximum profit. The dotted line shows that the actual peak (vertex) of the parabola lies slightly to the right of and above the uppermost point shown. Students' drawings may not be precise enough to detect this distinction.

Including the data from the final four columns in Problem #1



The four dots toward the right of the graph show data based on the final four columns in Problem #1. The points do not lie on the parabola because the demand function for their inputs does not fit the linear pattern of the others. More specifically, since the demand corresponding to these points is less than it would have been if the demand pattern had remained linear, the profit decreases. Thus, the points fall below the parabola.

Notes: The coordinates of the points are

(4.75, 607.5) (5.00, 400) (5.25, 110) (5.30, 0)

The calculations for the ordered pairs are

$$(4.75 - 2.50)270 = 607.5 \qquad (5.00 - 2.50)160 = 400$$
$$(5.25 - 2.50)40 = 110 \qquad (5.30 - 2.50)0 = 0$$

Stage 2

In Stage 2, students begin to analyze Latrice's situation more formally. In the process, they make many connections between equations and graphs of quadratic functions.

The first time around, you may be most comfortable using these problems to consolidate and apply previously learned concepts. However, some problems work well as vehicles to *teach* the concepts! This requires students to do more of the thinking and to develop many of the ideas themselves.

Problem #3

Students should probably be comfortable factoring trinomials before attempting this problem, but they do *not* necessarily need prior experience using factoring to solve quadratic equations.

Problem #4

Students may learn a lot from this problem by working on it *before* learning a procedure for completing the square. Be aware that this approach will be more challenging for students and will require more time.

Problem #5

Students should probably have some experience with the quadratic formula before working on this problem.

What students should know

- Understand concepts and prerequisite skills from Stage 1.
- Have experience with the quadratic formula (Problem #5 only).

What students will learn

- Discover and/or apply techniques for solving quadratic equations by factoring.
- Experience a meaningful context for the zero-product property (if $ab = 0$, then $a = 0$ or $b = 0$).
- Discover and/or apply a procedure for completing the square on a quadratic expression.
- Interpret the vertex form for a quadratic expression.
- Discover and/or apply methods for writing quadratic expressions in vertex form.
- Analyze relationships between the quadratic formula and the graph of a parabola.

Thinking Prompt #3

What do you notice? What do you wonder?

You can often answer questions more efficiently by analyzing and solving algebraic equations.

Projections

Price per card dollars	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50
Demand number per year	1000	920	840	760	680	600	520	440	360

Cost for making each card: $2.50

Problem #3

Projections

Price per card dollars	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50
Demand number per year	1000	920	840	760	680	600	520	440	360

Cost for making each card: $2.50

Directions

- Rewrite your price-profit equation from Problem #2.
- Write your quadratic expression in factored form (if you have not yet done so).
- Show how to use your factored-form expression to find the two prices at which the profit would be 0.
- Describe the relationship between these two prices and the optimum price. Explain your thinking.
- Explain how your responses to these questions would be affected if you included the data from the final four columns in Problem #1.

Conversation Starters for #3

Thinking Prompt

I notice that I can use my profit equation to predict the profit for any price that fits the pattern in the table.

I wonder if I can use my profit equation from Problem #2 to help me find the optimum selling price.

The Problem

I notice that the terms in the profit expression have a common factor.

I wonder if it is important to factor this number out.

I wonder if it is important to factor –1 out from the expression.

I notice that my profit expression from Problem #2 is already factored!

The common factor of –20 may not have been removed, but there is not always a need to do so.

I notice that I get the same answers for the prices that have a profit of 0 whether or not I factor a constant out of the profit expression.

I notice that the symmetry of the graph helps me understand more about relationships between price and profit.

Once you know the profit for one price, you can quickly find another price that has the same profit (from the mirror image point on the opposite side of the parabola).

Sample Solutions for #3

The price-profit equation from Problem #2

$$F(P) = -320P^2 + 2600P - 4500$$

The expression is in the standard form $aP^2 + bP + c$. Students' expressions may be written in other forms.

The quadratic expression in factored form

Students may obtain different answers depending on how they choose to handle the constant factor. If they factor out -20, the result is

$$F(P) = -20(2P - 5)(8P - 45)$$

Some students may notice that an equation they wrote in Problem #2 is already in factored form (if they do not worry about factoring out a constant)!

$$F(P) = (P - 2.50)(-320P + 1800)$$

The two prices at which the profit would be 0

The profit will equal 0 if either factor is equal to 0.

(1) $P - 2.50 = 0$ or $2P - 5 = 0$ will lead to $P = 2.50$
(2) $-320P + 1800 = 0$ or $8P - 45 = 0$ will lead to $P = 5.625$

If the demand function were linear throughout, the two prices having a profit of 0 would be \$2.50 and \$5.63. Notice that these are the two places where the dotted-line parabola (in the Solutions for Problem #2) crosses the P-axis.

A price of \$2.50 per card results in zero profit, because the cost to produce the card is equal to its selling price.

A price of \$5.63 per card would result in zero profit, because no one would buy the card at this price. (Again, remember that this actually occurs for prices as low as \$5.30 according to the original table in Problem #1. By applying the formulas above to larger values of P, we are imagining what would happen if the demand function were to remain linear throughout the table.)

Relationship between these two prices and the price that maximizes profit

The price that produces the maximum profit ($4.06) is the average (mean) of the two prices that result in 0 profit!

$$\frac{2.50 + 5.625}{2} = 4.0625 \approx 4.06$$

This makes sense because the parabola is symmetric, meaning that the P-coordinate of the vertex lies exactly midway between the two P-intercepts of the parabola.

The effect of using the data from the final four columns in Problem #1

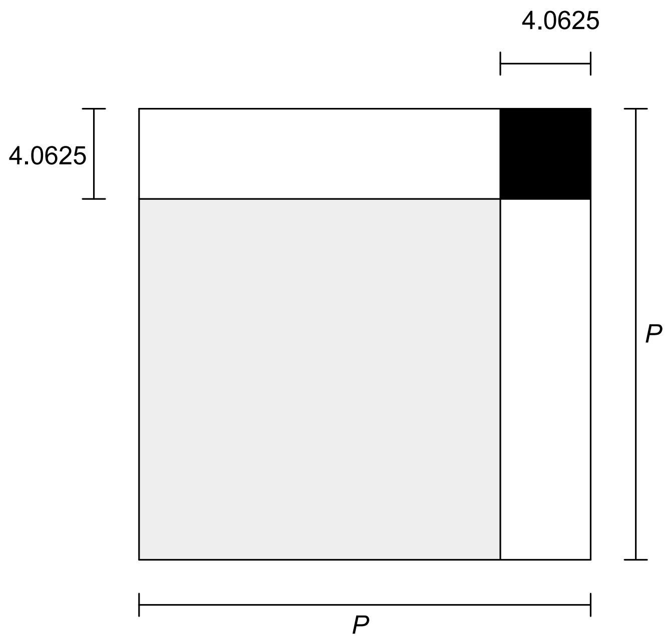
If you use the actual data from Problem #1, the optimum price is no longer the average of the prices that produce a profit of 0. The profit-price graph is no longer a parabola (and is not symmetric), because the four rightmost pairs in the original table do not fit the linear demand pattern.

In fact, if you had used the point (5.30, 0) as a P-intercept, the average would have been

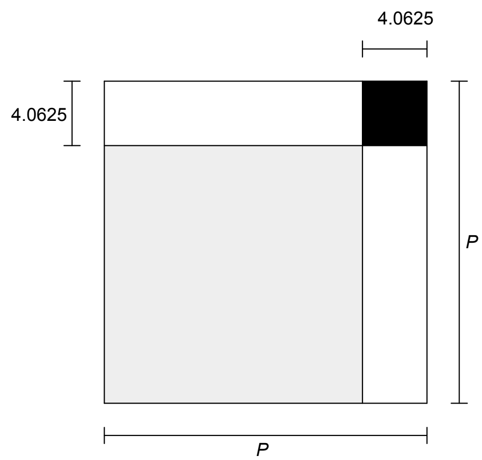
$$\frac{2.50 + 5.30}{2} = 3.90.$$

Thinking Prompt #4

What do you notice? What do you wonder?



Problem #4



Directions

- Write your quadratic expression in the standard form: $aP^2 + bP + c$.
- Factor the leading coefficient out of the quadratic and linear terms.
- Explain how (part of) your expression relates to the picture.
- Write your quadratic expression in *vertex form*: $a(P - h)^2 + k$.
- Explain how the vertex form of your expression relates to the optimum price.
- Explain why this happens.

Conversation Starters for #4

Thinking Prompt

I notice that P and 4.0625 are related to Latrice's business scenario.

I wonder why parts of the picture are shaded in grey and black.

I wonder why the entire square is decomposed into four parts.

I wonder what happens if I find the areas of some of the different regions in the picture.

The Problem

I notice that the area of the entire square is P^2 (which is part of the formula).

I notice that the number 4.0625 (in the picture) is half of the number 8.125 (that appears when I factor out the leading coefficient)!

I notice that there are multiple ways to write expressions for the areas of the regions.

I wonder if some of the areas in the pictures match up with parts of my profit expression.

I notice that it helps to keep looking back and forth between the picture and the profit expression in order to make connections.

I wonder what the a, h, and k are in the vertex form.

I notice that the $P - h$ in the vertex form expression matches the $P - 4.0625$ in the picture, which is the side length of the grey square.

I notice a messy expression at the end of my formula that I can write as a single number.

I notice that this number is the maximum profit!

I notice that $-320(P - 4.0625)^2$ in the vertex form expression can never be positive.

I notice that h and k in my expression are the coordinates of the vertex.

Hence the name "vertex form!"

Sample Solutions for #4

The quadratic expression in standard form

$$-320P^2 + 2600P - 4500$$

Factoring the leading coefficient out of the quadratic and linear terms

$$-320(P^2 - 8.125P) - 4500$$

How the expression relates to the picture

The area of the grey region is

$$(P - 4.0625)^2 \qquad \text{(Expression 1)}$$

which is not quite the same as the expression

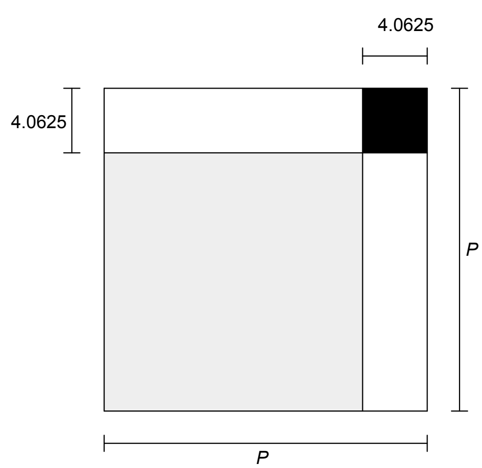
$$P^2 - 8.125P \qquad \text{(Expression 2)}$$

in parentheses. However, we can make them the same with a small adjustment to Expression 2. Some students may begin by observing that 4.0625 is half of 8.125.

The area of the entire square is P^2. Suppose that, in order to find another way to write the area of the grey region, you remove the two rectangles at the top and the right (the ones with areas of $4.0625P$). Subtracting these two areas from the original area leaves

$$P^2 - 4.0626P - 4.0625P = P^2 - 8.125P.$$

This is Expression 2, but it is not yet quite the same as the area of the grey region, because you took away the area of the small square in the corner (shown here in black) twice!



To compensate, you simply add this area (which is 4.0625^2) back in, getting a result of

$$P^2 - 8.125P + 4.0625^2$$

for the area of the grey region.

Putting all of this together, you have found two equivalent expressions for the area of grey region:

$$(P - 4.0625)^2 = P^2 - 8.125P + 4.0625^2.$$

Of course, students may verify this relationship by multiplying out the expression on the left. The picture is simply a way to visualize it!

More importantly for our purposes, the picture illustrates the process of “completing the square.” Specifically, you can transform the expression

$$P^2 - 8.125P$$

into the square of a simple linear expression by adding an appropriately chosen constant. By reviewing the process closely, you will see that the constant (which is the area of the small black square) must always be the square of half the coefficient of P.

Writing the quadratic expression in vertex form.

The vertex form for the expression is $-320(P - 4.0625)^2 + 781.25$.

Thinking process

Vertex form for this situation looks like

$$a(P - h)^2 + k$$

where a, h, and k are numbers that need to be determined, and P represents the price of a card. The key to writing the quadratic expression in this form is to complete the square. Thus, much of the work is already complete!

(1) $-320P^2 + 2600P - 4500 =$
(2) $-320(P^2 - 8.125P) - 4500 =$
(3) $-320(P^2 - 8.125P + 4.0625^2) - 4500 + 320(4.0625^2) =$
(4) $-320(P - 4.0625)^2 + 781.25$

- Step 2 shows the process of factoring the leading coefficient out from the quadratic and linear terms.
- Step 3 involves more thinking. You add the appropriate constant to the expression in parentheses (to make it a perfect square). This process adds $-320(4.0625)^2$ to the expression, so you must compensate by adding the opposite of this value to the constant on the right.

- Step 4 involves rewriting the trinomial in parentheses as the square of a linear expression (as demonstrated in the picture) and calculating the new value of the constant term.

How vertex form relates to the optimum price

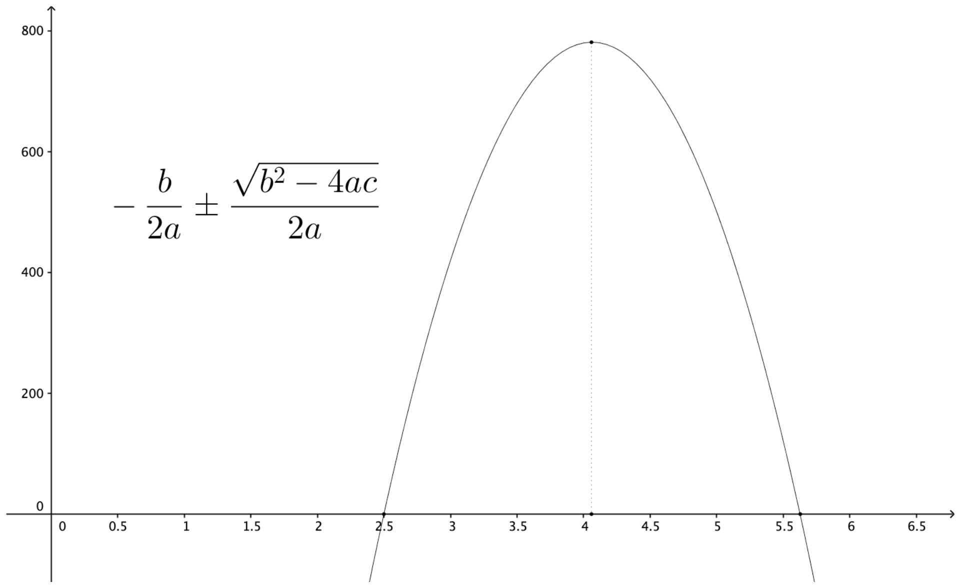
The constants, 4.0625 and 781.25, in the vertex form expression are the optimum price and the profit for that price, respectively. That is, you can read them directly from the expression!

Students can reason this out before being taught explicitly about the significance of vertex form. Since -320 is negative and $(P-4.0625)^2$ is never negative, the greatest possible value of $-320(P-4.0625)^2$ is 0, which occurs when the quantity being squared is equal to 0. Therefore, 781.25 is the greatest possible value of $-320(P-4.0625)^2+781.25$, and this happens when $P-4.0625=0$; that is, when $P=4.0625$.

All of this explains the name *vertex form.* The ordered pair (h,k)—in this case, $(4.0625{,}781.25)$—is the vertex of the parabola.

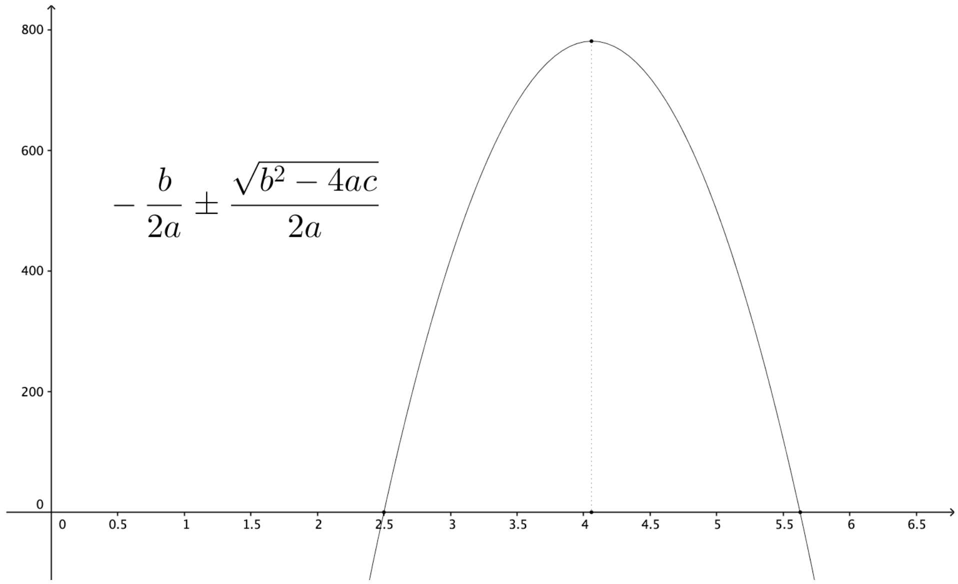
Thinking Prompt #5

What do you notice? What do you wonder?



$$-\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Problem #5



Directions

- Set your quadratic expression (in standard form) equal to 0, and identify the values of *a*, *b*, and *c.*
- Find the values of $-\frac{b}{2a}$ and $\left|\frac{\sqrt{b^2-4ac}}{2a}\right|$.
- Show where or how the expressions $-\frac{b}{2a}$ and $\left|\frac{\sqrt{b^2-4ac}}{2a}\right|$ fit on the graph.
- Use the graph to explain how the two expressions relate to the solutions of your quadratic equation.
- Explain the significance of the expression $-\frac{b}{2a}$ for Latrice's business.

Conversation Starters for #5

Thinking Prompt

I notice that the parabola is the graph of the profit function from this project.

I wonder why the quadratic formula and the parabola are shown together.

I wonder why the quadratic formula is split into two parts.

I wonder what happens if I calculate the value of each part separately.

The problem

I wonder why there is an absolute value symbol around the square root expression.

> Without the parentheses, the value of the expression may be either positive or negative, depending on the sign of a. The absolute value ensures that its value will always be non-negative and can thus represent a distance. Including the absolute value does not affect the solutions given by the quadratic formula because of the "$\pm$" symbol.

I notice that the value of $-\frac{b}{2a}$ is the same as the optimum price!

I wonder if $-\frac{b}{2a}$ will always be the x-coordinate of the vertex (and if so, why).

I wonder if adding and subtracting the values of the two expressions, $-\frac{b}{2a}$ and $\left|\frac{\sqrt{b^2-4ac}}{2a}\right|$, will help me to see more connections to the graph.

I wonder if (or how) I could have found the profit formula directly from the graph if I had not already calculated it.

Sample Solutions for #5

Identifying the values of a, b, and c

$$-320P^2 + 2600P - 4500 = 0$$

The values of a, b, and c are $-320, 2600$, and -4500, respectively.

Calculating values for the two expressions

The values of $-\frac{b}{2a}$ and $\left|\frac{\sqrt{b^2-4ac}}{2a}\right|$ are 4.0625 and 1.5625, respectively.

$$-\frac{b}{2a} = -\frac{2600}{2(-320)} = 4.0625$$

$$\left|\frac{\sqrt{b^2 - 4ac}}{2a}\right| = \left|\frac{\sqrt{(2600)^2 - 4(-320)(-4500)}}{2(-320)}\right|$$

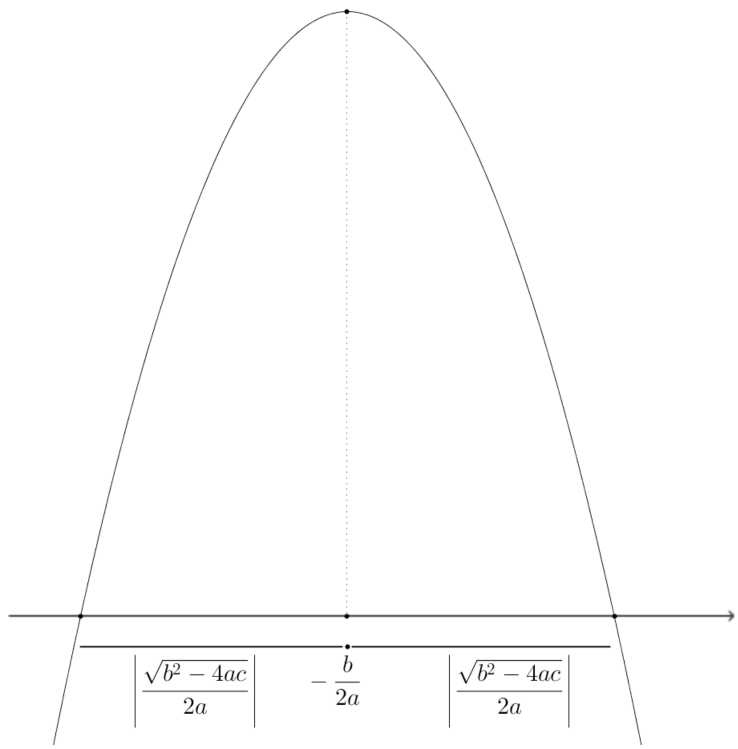
$$= \left|\frac{\sqrt{6{,}760{,}000 - 5{,}760{,}000}}{-640}\right| = \left|\frac{\sqrt{1{,}000{,}000}}{-640}\right| = \frac{1000}{640} = 1.5625$$

Separating the expression in the quadratic formula into two fractions helps students to explore a geometric interpretation for each part.

The relationship between these expressions and the graph

$-\frac{b}{2a}$ is the P-coordinate of the vertex of the graph (and $P = -\frac{b}{2a}$ is the equation of the axis of the parabola).

$\left|\frac{\sqrt{b^2-4ac}}{2a}\right|$ is the distance between the x-coordinate of the vertex and each of the P-intercepts of the parabola. (In fact, the expression $\left|\frac{\sqrt{b^2-4ac}}{2a}\right|$ contains the absolute value in order to emphasize its interpretation as a distance.)



How the expressions relate to the solutions of $-320P^2 + 2600P - 4500 = 0$

To find the two solutions, begin at $-\frac{b}{2a}$; then add $\left|\frac{\sqrt{b^2-4ac}}{2a}\right|$ to obtain the greater solution and subtract $\left|\frac{\sqrt{b^2-4ac}}{2a}\right|$ to obtain the lesser solution.

In this particular case,

$$4.0625 + 1.5625 = 5.625$$
$$4.0625 - 1.5625 = 2.50$$

Note: Students are already familiar with these calculations from their study of the quadratic formula. The purpose of this question is to connect this knowledge to their observations about the graph.

The significance of $-\frac{b}{2a}$ *for Latrice's business*

$-\frac{b}{2a}$ represents the selling price that gives maximum profit.

Stage 3

In Problem #6, students generalize their results from earlier in the exploration by representing formerly constant values as variables. The end product is a set of two algebraic expressions that solves the optimization problem for arbitrary values of the problem's original parameters.

Students may do the two tasks in the directions for Problem #6 in either order.

What students should know

- Understand concepts and complete the tasks from Stages 1 and 2.

What students will learn

- Apply and extend skills and concepts from earlier in the exploration by carrying out complex computations with variables in place of numeric parameters.
- Interpret the results of these computations in a real-world context.

Thinking Prompt #6

What do you notice? What do you wonder?

Demand-price equation: $D = mP + n$

Cost to produce one item: C

Problem #6

Demand-price equation: $D = mP + n$
Cost to produce one item: C

Directions

- Given the cost and the demand-price equation above, prove that the optimum price is always the mean of the two prices that result in zero revenue.
- Find expressions in terms of m, n, and C for the optimum selling price and its annual profit.

Conversation Starters for #6

Thinking Prompt

I notice that I have already seen this equation for profit.

I notice that the cost and the profit equation are expressied in a *general* form.

They refer to demand-profit situations that may involve *any* (realistic) numbers for m, n, and C.

I wonder what new questions I can ask about the formulas.

I wonder if I can apply what I learned in the Stage 2 problems to these equations.

The problem

I notice that I can apply the same processes to the general equations in this problem that I used in previous problems when I had specific numbers to work with.

I notice that the graphical visualization in Problem #5 makes it easier to think of strategies for working with the general expressions in this problem.

I notice that there are multiple options for finding an equation for the maximum profit.

I notice that when I complete the square on the general profit expression, some parts of the result remind me of the quadratic formula.

This is not a coincidence, because the quadratic formula comes from completing the square on $ax^2 + bx + c$! The main difference in this case is that a, b, and c are replaced by more complicated expressions that are specific to the context of demand and price.

I notice that I can test my formulas by seeing if they give the same results for the optimum price and profit that I found earlier in the problem.

Sample Solutions for #6

Proof that the optimum price is the mean of the prices that give zero profit

Applying the equation given in the problem

Given the equation $D = mP + n$ for the demand and the variable C for the cost of one item, a factored form of the profit function is:

$$F(P) = (P - C)D$$
$$F(P) = (P - C)(mP + n)$$

The prices that give a profit of 0 are the P-intercepts, which you may find by setting each factor equal to 0.

$$P - C = 0 \quad \rightarrow \quad P = C$$
$$mP + n = 0 \quad \rightarrow \quad P = -\frac{n}{m}.$$

The mean of the P-intercepts is

$$\frac{1}{2}\left(C + -\frac{n}{m}\right) = \frac{mC - n}{2m}$$

The optimum price is the P-coordinate of the vertex. Having discovered in Problem #5 that its value is given by $-\frac{b}{2a}$, many students may calculate it directly as

$$-\frac{b}{2a} = -\frac{n - mC}{2m} = \frac{mC - n}{2m}$$

which is equal to the mean of the P-intercepts calculated above!

Others may complete the square in order to write the profit equation in vertex form (a rather messy calculation).

$$F(P) = mP^2 + (n - mC)P - nC$$
$$F(P) = m(P^2 + \frac{n-mC}{m}P) - nC$$
$$F(P) = m\left(P^2 - \frac{mC-n}{m}P + \left(\frac{mC-n}{2m}\right)^2\right) - nC - m\left(\frac{mC-n}{2m}\right)^2$$
$$F(P) = m\left(P - \frac{mC - n}{2m}\right)^2 + \left(-nC - \frac{(mC - n)^2}{4m}\right)$$

This equation is in vertex form: $m(P - h)^2 + k$. The P-coordinate of the vertex is $h = \frac{mC-n}{2m}$.

Some students may prefer to prove the relationship using other ideas from this project. While the following justifications do not make explicit use of the demand or cost equations as suggested in the directions, they are sound arguments *when the demand equation is linear*.

An informal geometric approach

If the demand equation is linear, then the profit function is quadratic (as shown earlier). The graph of a quadratic function is a parabola, which is symmetric about a vertical line through its vertex. Thus, the P-coordinate of the vertex is the midpoint of the segment joining the P-intercepts of the parabola.

The conclusion follows, because (1) the P-coordinate of the vertex is the optimum selling price, (2) the P-intercepts are the prices that give zero profit, and (3) the mean of two numbers is exactly midway between them.

Using the quadratic formula to formalize the geometric argument

Some students may explicitly calculate the mean of the two solutions directly from the quadratic formula. The messy appearance of the calculation makes the idea look much more complicated than it really is!

$$\frac{1}{2}\left[\left(-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)+\left(-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)\right]=$$
$$\frac{1}{2}\left[\left(-\frac{b}{2a}+-\frac{b}{2a}\right)+\left(\frac{\sqrt{b^2-4ac}}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)\right]=$$
$$\frac{1}{2}\left[\left(-\frac{2b}{2a}\right)+0\right]=\frac{1}{2}\left(-\frac{b}{a}\right)=-\frac{b}{2a}$$

Expressions for the optimum selling price and its annual profit

Students who have found the vertex form, $m(P-h)^2+k$, of the profit expression may read the values of h and k directly from it!

Optimum selling price:

$$h=\frac{mC-n}{2m}$$

Optimum profit:

$$k=-nC-\frac{(mC-n)^2}{4m}$$

Others may substitute the optimum selling price into the profit expression.

$$F(P) = mP^2 + (n - mC)P - nC$$
$$F\left(\frac{mC - n}{2m}\right) = m\left(\frac{mC - n}{2m}\right)^2 + (n - mC)\left(\frac{mC - n}{2m}\right) - nC$$

They may leave this expression as it is or simplify it somewhat. For example:

$$m\frac{(mC - n)^2}{4m^2} - (mC - n)\left(\frac{mC - n}{2m}\right) - nC$$
$$= \frac{(mC - n)^2}{4m} - \frac{(mC - n)^2}{2m} - nC$$
$$= \frac{(mC - n)^2}{4m} - \frac{2(mC - n)^2}{4m} - nC$$
$$= -\frac{(mC - n)^2}{4m} - nC$$

which is equivalent to the expression obtained by completing the square.

Students may verify that these formulas give the correct results for the particular numbers in this context. Remembering that $D = -320P + 1800$ and $C = 2.50$:

Optimum selling price:

$$\frac{mC - n}{2m} = \frac{-320(2.50) - 1800}{2(-320)}$$

$$= \frac{-800 - 1800}{-640} = \frac{-2600}{-640} = 4.0625$$

Optimum profit:

$$-nC - \frac{(mC - n)^2}{4m} =$$
$$= -(1800)(2.50) - \frac{(-320 \cdot 2.50 - 1800)^2}{4(-320)}$$

$$= -4500 - \frac{(-800 - 1800)^2}{-1280} = -4500 - \frac{(2600)^2}{-1280}$$

$$= -4500 + \frac{6{,}760{,}000}{1280} =$$

$$= -4500 + 5281.25 = 781.25$$